

# CRITERIA FOR THE DENSITY OF THE GRAPH OF THE ENTROPY MAP RESTRICTED TO ERGODIC STATES

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**ABSTRACT.** We consider a non-uniquely ergodic dynamical system given by a  $\mathbb{Z}^l$ -action (or  $(\mathbb{N} \cup \{0\})^l$ -action)  $\tau$  on a non-empty compact metrisable space  $\Omega$ , for some  $l \in \mathbb{N}$ . Let (D) denote the following property: The graph of the restriction of the entropy map  $h^\tau$  to the set of ergodic states is dense in the graph of  $h^\tau$ . We assume that  $h^\tau$  is finite and upper semi-continuous. We give several criteria in order that (D) holds, each of which is stated in terms of a basic notion: Gateaux differentiability of the pressure map  $P^\tau$  on some sets dense in the space  $C(\Omega)$  of real-valued continuous functions on  $\Omega$ , level-2 large deviation principle, level-1 large deviation principle, convexity properties of some maps on  $\mathbb{R}^n$  for all  $n \in \mathbb{N}$ . The one involving the Gateaux differentiability of  $P^\tau$  is of particular relevance in the context of large deviations since it establishes a clear comparison with another well-known sufficient condition: We show that for each non-empty  $\sigma$ -compact subset  $\Sigma$  of  $C(\Omega)$ , (D) is equivalent to the existence of an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that  $f+g$  has a unique equilibrium state for all  $(f, g) \in \Sigma \times V \setminus \{0\}$ ; any Schauder basis  $(f_n)$  of  $C(\Omega)$  whose linear span contains  $\Sigma$  admits an arbitrary small perturbation  $(h_n)$  so that one can take  $V = \text{span}(\{f_n + h_n : n \in \mathbb{N}\})$ . Taking  $\Sigma = \{0\}$ , the existence of an infinite dimensional vector space dense in  $C(\Omega)$  constituted by functions admitting a unique equilibrium state is equivalent to (D) together with the uniqueness of measure of maximal entropy.

## 1. INTRODUCTION

Let  $(\Omega, \tau)$  be a dynamical system in the sense of [17] (*i.e.*  $\Omega$  is a non-empty compact metrizable space and  $\tau$  an action of  $\mathbb{Z}^l$  (resp.  $(\mathbb{N} \cup \{0\})^l$ ) on  $\Omega$  for some  $l \in \mathbb{N}$ ). Let  $C(\Omega)$ ,  $\mathcal{M}(\Omega)$ ,  $\mathcal{M}^\tau(\Omega)$ ,  $\mathcal{E}^\tau(\Omega)$ ,  $h^\tau$ ,  $P^\tau$  denote respectively the set of real-valued continuous functions on  $\Omega$  endowed with the uniform topology, Borel probability measures on  $\Omega$  endowed with the weak-\* topology,  $\tau$ -invariant elements of  $\mathcal{M}(\Omega)$ , ergodic elements of  $\mathcal{M}^\tau(\Omega)$ , measure-theoretic entropy and pressure maps. We assume that  $\mathcal{M}^\tau(\Omega)$  is not a singleton and  $h^\tau$  is finite and upper semi-continuous.

In some basic dynamical systems as above (*e.g.* full shifts) the set  $\mathcal{M}^\tau(\Omega)$  fulfils a fundamental density property: Not only  $\mathcal{E}^\tau(\Omega)$  is dense in  $\mathcal{M}^\tau(\Omega)$  (*i.e.*  $\mathcal{M}^\tau(\Omega)$  is the Poulsen simplex) but the set  $\{\mu \in \mathcal{E}^\tau(\Omega) : h^\tau(\mu) > r\}$  is dense in the set  $\{\mu \in \mathcal{M}^\tau(\Omega) : h^\tau(\mu) > r\}$  for every real  $r$ ; thanks to the upper semi-continuity of  $h^\tau$ , this is equivalent to the density of the graph of the restricted map  $h^\tau|_{\mathcal{E}^\tau(\Omega)}$  in the graph of  $h^\tau$ ; the importance of this

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2000 *Mathematics Subject Classification.* Primary: 37D35; Secondary: 37A50, 37D25, 60F10.

<sup>†</sup> Partially supported by FONDECYT grant 1120493.

property has long been recognized, *cf.* [9], [10], [19] (in particular, it implies the nowhere Frechet differentiability of  $P^\tau$ ); following [9] let us denote it by (D). Since a measure on  $\Omega$  is ergodic if and only if it is the unique equilibrium state for some element in  $C(\Omega)$  ([14]), (D) is also equivalent to Property 5.1 of [3] which turns out to be sufficient to ensure the large deviation principle for any net  $(\nu_\alpha)$  of Borel probability measures on  $\mathcal{M}(\Omega)$  and any net  $(t_\alpha)$  of positive real numbers converging to zero fulfilling for some (arbitrary)  $f \in C(\Omega)$ ,

$$\forall g \in C(\Omega), \quad \lim_{\alpha} t_{\alpha} \log \int_{\mathcal{M}(\Omega)} e^{t_{\alpha}^{-1} \widehat{g}(\mu)} \nu_{\alpha}(d\mu) = P^{\tau}(f + g) - P^{\tau}(f), \quad (1)$$

where  $\widehat{g}(\mu) = \int_{\Omega} g(\xi) \mu(d\xi)$  ([3], Theorem 5.2). Another well-known sufficient condition to get the large deviation principle for nets  $(\nu_{\alpha}, t_{\alpha})$  fulfilling (1) is the existence of a vector space  $V$  dense in  $C(\Omega)$  such that  $f + g$  has a unique equilibrium state for all  $g \in V$  ([12], [2]); note that by taking  $g = 0$  this implies the uniqueness of equilibrium for  $f$ , whereas (D) does not impose any conditions on  $f$ .

A basic problem is to compare the two above conditions: Does one imply the other? Are they equivalent? If not, which extra hypotheses have to be added to get an equivalence? We can also compare them with the large deviation property: Do the large deviation principles imply one (or both) of these conditions? If not, do exist simple extra hypotheses on the rate function in order to get an equivalence? The same questions arise for the net  $((\widehat{f}_1, \dots, \widehat{f}_n)[\nu_{\alpha}])$  image of  $(\nu_{\alpha})$  by the map  $(\widehat{f}_1, \dots, \widehat{f}_n)$  for any  $((f_1, \dots, f_n), n) \in C(\Omega)^n \times \mathbb{N}$  (and more generally for any net of Borel probability measures on  $\mathbb{R}^n$  admitting the same limiting log-moment generating function as  $((\widehat{f}_1, \dots, \widehat{f}_n)[\nu_{\alpha}])$ ). As long as one is only concerned by (D), more than the nature of the net satisfying the large deviation principle, the most relevant object is the rate function; more specifically, some fine convexity properties of the rate function play a major role. This leads us to consider convexity properties of some maps involving only the restriction of  $P^\tau$  to finite dimensional spaces (or equivalently, its dual version with  $h^\tau$ ) as a new element of comparison.

In this paper we answer to the preceding questions, showing that the five above properties (*i.e.* (D), Gateaux differentiability of  $P^\tau$ , large deviation principle on  $\mathcal{M}(\Omega)$ , large deviation principle on  $\mathbb{R}^n$  for all  $n \in \mathbb{N}$ , convexity properties of some maps on  $\mathbb{R}^n$  for all  $n \in \mathbb{N}$ ) are in fact equivalent once specified how they take place (Theorem 1); in particular, we obtain a plain and simple comparison between the two above mentioned general sufficient conditions to get the large deviation principle for nets  $(\nu_{\alpha}, t_{\alpha})$  as in (1): (D) is equivalent to the Gateaux differentiability of  $P^\tau$  on an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  possibly excepting zero; furthermore, for each Schauder basis  $(f_n)$  of  $C(\Omega)$  and for each sequence  $(\varepsilon_n)$  of positive real numbers converging to zero, there is a sequence  $(h_n)$  in  $C(\Omega) \setminus \{0\}$  with  $\|h_n\| \leq \varepsilon_n$  so that one can take  $V = \text{span}(\{f_n + h_n : n \in \mathbb{N}\})$ ; when such a space  $V$  is obtained, a Schauder basis may be used to get another vector space linearly independent from  $V$  whose direct sum with  $V$  fulfils the same property as  $V$ ; iterating this process gives rise to a new criterion for the validity of (D) (Theorem 2). When there is a unique measure of maximal entropy, the above conditions can be greatly simplified

(Corollary 1). As a by-product, the large deviation results of [2] are both generalized and strengthened (Example 1).

In the next section we review some basic notions of thermodynamic formalism, large deviation theory and convex analysis. The results are stated in Section 3. The proofs are given in Section 4. The proof of the main theorem uses a particular case of two results of Israel and Phelps ([9]) that we recall in Appendix A.

## 2. PRELIMINARIES

**2.1. Thermodynamic formalism.** Let  $(\Omega, \tau)$  be a dynamical system as in §1. Put  $\Lambda(a) = \{(x_1, \dots, x_l) \in (\mathbb{N} \cup \{0\})^l : x_i < a_i, 1 \leq i \leq l\}$  and let  $|\Lambda(a)|$  denote the cardinality of  $\Lambda(a)$  for all  $a \in \mathbb{N}^l$ . For each  $\varepsilon > 0$  and for each  $a \in \mathbb{N}^l$  let  $\Omega_{\varepsilon, a}$  be a maximal  $(\varepsilon, \Lambda(a))$ -separated set. Order  $\mathbb{N}^l$  lexicographically. Recall that  $P^\tau(g)$  is defined for each  $g \in C(\Omega)$  by

$$P^\tau(g) = \lim_{\varepsilon \rightarrow 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon, a}} e^{\sum_{x \in \Lambda(a)} g(\tau^x \xi)}, \quad (2)$$

and fulfills

$$P^\tau(g) = \lim_{\varepsilon \rightarrow 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon, a}} e^{\sum_{x \in \Lambda(a)} g(\tau^x \xi)} = \sup_{\mu \in \mathcal{M}^\tau(\Omega)} \{\mu(g) + h^\tau(\mu)\}. \quad (3)$$

([17], §6.7, §6.12 and Exercise 2 p. 119 for  $\mathbb{Z}^l$ -action, §6.18 for  $(\mathbb{N} \cup \{0\})^l$ -action). Since  $\mathcal{M}^\tau(\Omega)$  is compact and  $h^\tau$  is finite and upper semi-continuous, the above supremum is a maximum, and each element realizing this maximum is called an equilibrium state for  $g$ . The map  $P^\tau$  is finite convex and continuous on  $C(\Omega)$  ([17], §6.8 and §6.18). The right hand side of (2) may be called the topological pressure versus the variational pressure appearing in the right hand side of the last equality in (3); the equality between both quantities is known as the variational principle. The map  $h^\tau$  is affine; the set  $\mathcal{M}^\tau(\Omega)$  is a non-empty metrizable Choquet simplex and  $\mathcal{E}^\tau(\Omega)$  is the set of extreme points of  $\mathcal{M}^\tau(\Omega)$  ([17], §6.1, §6.5 and §6.18).

A sequence  $(f_n)$  in  $C(\Omega)$  is a Schauder basis of  $C(\Omega)$  if for each  $f \in C(\Omega)$  there exists a unique sequence  $(\lambda_n(f))$  of real numbers such that  $\lim_n \|f - \sum_{k=1}^n \lambda_k(f) f_k\| = 0$ . It is known that  $C(\Omega)$  admits a Schauder basis ([18], Theorem 4.4.13 and Notes pp. 8-9); furthermore, each vector space dense in  $C(\Omega)$  contains a Schauder basis of  $C(\Omega)$  ([18], Corollary 1.1.9). Each Schauder basis  $(f_n)$  of  $C(\Omega)$  fulfils

$$\forall n \in \mathbb{N}, \quad \sup_{f \in C(\Omega), \|f\| \leq 1} |\lambda_n(f)| < +\infty$$

([18], Proposition 1.1.6). We will use the following result: For each Schauder basis  $(f_n)$  of  $C(\Omega)$  and for each sequence  $(h_n)$  in  $C(\Omega)$  fulfilling

$$\sum_{n=1}^{+\infty} \left( \sup_{f \in C(\Omega), \|f\| \leq 1} |\lambda_n(f)| \right) \|h_n\| < 1,$$

the sequence  $(f_n + h_n)$  is a Schauder basis of  $C(\Omega)$  ([18], Theorem 1.1.8).

**2.2. Convex analysis.** Let  $X$  be a Hausdorff real topological vector space and let  $X^*$  be the topological dual of  $X$  endowed with the weak-\* topology. Let  $A$  be a nonempty convex subset of  $X$  and let  $I$  be a  $]-\infty, +\infty]$ -valued function on  $A$ . The function  $I$  is convex if

$$\forall (x, y, \lambda) \in A^2 \times [0, 1], \quad I(\lambda x + (1 - \lambda)y) \leq \lambda I(x) + (1 - \lambda)I(y).$$

The set of all  $x \in A$  such that  $I(x) \in \mathbb{R}$  is called the effective domain of  $I$ . The function  $I$  is proper if the effective domain of  $I$  is nonempty; in this case, for each convex subset  $C$  of the effective domain of  $I$ ,  $I$  is said to be strictly convex on  $C$  if

$$I(\lambda x + (1 - \lambda)y) < \lambda I(x) + (1 - \lambda)I(y)$$

for all  $(x, y, \lambda) \in C^2 \times ]0, 1[$  with  $x \neq y$ .

Let us assume furthermore that  $A = X$ . The Legendre-Fenchel transform (also called convex conjugate)  $I^*$  of  $I$  is the function defined on  $X^*$  by

$$\forall u \in X^*, \quad I^*(u) = \sup_{x \in X} \{u(x) - I(x)\};$$

note that  $I^*$  is a proper convex function on  $X^*$  when  $I$  is proper. An element  $u \in X^*$  is a subgradient of  $I$  at  $x \in X$  if

$$\forall y \in X, \quad I(y) \geq I(x) + u(y - x);$$

note that when  $I$  is proper the above inequality implies that  $x$  belongs to the effective domain of  $I$ . An element  $u \in X^*$  is a subgradient of  $I$  at  $x \in X$  if and only if one of the following conditions holds:

- $I^*(u) + I(x) \leq u(x)$ ;
- $I^*(u) + I(x) = u(x)$ .

If  $I$  is lower semi-continuous, then  $u \in X^*$  is a subgradient of  $I$  at  $x \in X$  if and only if  $x \in X$  is a subgradient of  $I^*$  at  $u \in X^*$  ([7], Corollary 5.2). For each  $(x, y) \in X^2$  we put

$$dI(x; y) = \lim_{\varepsilon \rightarrow 0^+} \frac{I(x + \varepsilon y) - I(x)}{\varepsilon};$$

$dI(x; y)$  is a well-defined element of the extended real line (by convexity) and called the directional derivative of  $I$  at  $x$  in the direction  $y$ ;  $I$  is Gateaux differentiable at  $x$  if there exists  $u \in X^*$  such that

$$\forall y \in X, \quad dI(x; y) = u(y);$$

such an element  $u$  is unique and called the Gateaux differential of  $I$  at  $x$ . When furthermore  $X$  is locally convex we have  $I = I^{**}|_X$  ([7], Proposition 3.1 and Proposition 4.1) and the two following results hold for all  $x \in X$ : If  $I(x) \in \mathbb{R}$  and  $I$  is continuous at  $x$ , then  $I$  is Gateaux differentiable at  $x$  if and only if  $I$  has a unique subgradient at  $x$ ; in this case, this subgradient is the Gateaux differential of  $I$  at  $x$  ([7], Proposition 5.3).

The above notions will be applied in a finite as well as infinite dimensional setting; in this latter case we shall consider  $X = C(\Omega)$  and  $I = P^\tau$ . Some results recalled in §2.1 may then be rephrased in terms of convex analysis: For each  $g \in X$ , an element  $\mu \in X^*$  is a subgradient of  $P^\tau$  at  $g$  if and only if  $\mu$  is an equilibrium state for  $g$ ; in particular,  $g$  admits a unique equilibrium state if and only if  $P^\tau$  is Gateaux differentiable at  $g$ . The

variational principle asserts that the entropy map  $h^\tau$  is the restriction to  $\mathcal{M}^\tau(\Omega)$  of the Legendre-Fenchel transform of  $P^\tau$ .

In the finite dimensional setting we will need the notion of essential differentiability. Let  $n \in \mathbb{N}$ . We assume that  $X = \mathbb{R}^n$  and  $I$  is proper. Let  $\text{dom } \delta I$  denote the set of points where  $I$  has a subgradient; note that  $\text{dom } \delta I \neq \emptyset$  when the effective domain is not a singleton (*cf.* [16], Theorem 23.4). Then,  $I$  is said to be essentially strictly convex if  $I$  is strictly convex on every convex subset of  $\text{dom } \delta I$ . The function  $I$  may be essentially strictly convex but not strictly convex on its effective domain; on the other hand,  $I$  may be strictly convex on the relative interior of its effective domain, but not essentially strictly convex (*cf.* [16]). It is known that when  $I$  is lower semi-continuous and  $I^*$  has effective domain  $\mathbb{R}^n$ , then  $I$  is essentially strictly convex if and only if  $I^*$  is differentiable on  $\mathbb{R}^n$  ([16], Theorem 26.3).

**2.3. Large deviations.** Let  $(\nu_\alpha, t_\alpha)$  be a net where  $\nu_\alpha$  is a Borel probability measure on a Hausdorff regular topological space  $X$ ,  $t_\alpha > 0$  and  $(t_\alpha)$  converges to zero. We say that  $(\nu_\alpha)$  satisfies a large deviation principle with powers  $(t_\alpha)$  if there exists a  $[0, +\infty]$ -valued lower semi-continuous function  $I$  on  $X$  such that

$$\limsup t_\alpha \log \nu_\alpha(F) \leq - \inf_{x \in F} I(x) \leq - \inf_{x \in G} I(x) \leq \liminf t_\alpha \log \nu_\alpha(G)$$

for all closed sets  $F \subset X$  and all open sets  $G \subset X$  with  $F \subset G$ ; such a function  $I$  is then unique and called the rate function.

Assume furthermore that  $X$  is a real topological vector space with topological dual  $X^*$  endowed with the weak-\* topology. The map  $\overline{L}$  defined on  $X^*$  by

$$\forall \lambda \in X^*, \quad \overline{L}(\lambda) = \limsup_{\alpha} t_\alpha \log \int_X e^{t_\alpha^{-1} \lambda(x)} \nu_\alpha(dx)$$

is called the generalized log-moment generating function (associated with  $(\nu_\alpha, t_\alpha)$ ); it is a  $]-\infty, +\infty]$ -valued proper convex function; when the above upper limit is a limit it is called the limiting log-moment generating function at  $\lambda$ .

The net  $(\nu_\alpha)$  is said to be exponentially tight with respect to  $(t_\alpha)$  if for each real  $M$  there exists a compact set  $K_M \subset X$  such that

$$\limsup_{\alpha} t_\alpha \log \nu_\alpha(X \setminus K_M) < M.$$

It is known that if  $(\nu_\alpha)$  is exponentially tight with respect to  $(t_\alpha)$  and  $\overline{L}$  is  $\mathbb{R}$ -valued, then  $\overline{L}$  is weak-\* lower semi-continuous; when furthermore  $X$  is locally convex and  $(\nu_\alpha)$  satisfies a large deviation principle with powers  $(t_\alpha)$  and convex rate function  $I$ , then  $I = \overline{L}^*$ , where  $\overline{L}^*$  is the Legendre-Fenchel transform of  $\overline{L}$  ([6], Theorem 4.5.10; [4], Corollary 2); in this case we have  $I^* = \overline{L}$  by the lower semi-continuity of  $\overline{L}$ . If  $X = \mathbb{R}^n$  for some  $n \in \mathbb{N}$  and the limiting log-moment generating function exists and is real-valued and differentiable on  $\mathbb{R}^n$ , then a Gärtner's theorem asserts that  $(\nu_\alpha)$  satisfies a large deviation principle with powers  $(t_\alpha)$  and rate function  $\overline{L}^*$  ([8]; [6], Theorem 2.3.6).

If  $(\nu_\alpha)$  satisfies a large deviation principle with powers  $(t_\alpha)$  and rate function  $I$ , then for each real-valued continuous function  $\lambda$  on  $X$  fulfilling

$$\lim_{M \rightarrow +\infty} \limsup_{\alpha} t_\alpha \log \int_{\{x \in X : \lambda(x) > M\}} e^{t_\alpha^{-1} \lambda(x)} \nu_\alpha(dx) = -\infty \quad (4)$$

the limiting log-moment generating function at  $\lambda$  exists and given by

$$\overline{L}(\lambda) = \sup_{x \in X} \{\lambda(x) - I(x)\}. \quad (5)$$

The above result is a particular case of the Varadhan's theorem whose original statement requires the compactness of the level sets  $\{x \in X : \lambda(x) \leq r\}$  for all  $r \in \mathbb{R}$  ([6], Theorem 4.3.1); this hypothesis has been removed in [5], Corollary 3.4. When  $I$  is convex and  $\lambda \in X^*$ , the equality (5) can be written as  $\overline{L}(\lambda) = I^*(\lambda)$ , where  $I^*$  denotes the Legendre-Fenchel transform of  $I$ .

Let  $\widetilde{\mathcal{M}}(\Omega)$  denote the set of signed Radon measures on  $\Omega$  endowed with the weak-\* topology. The above notions will be applied with  $X = \widetilde{\mathcal{M}}(\Omega)$ ,  $X = \mathcal{M}(\Omega)$  and  $X = \mathbb{R}^n$  for all  $n \in \mathbb{N}$ . Since  $\widetilde{\mathcal{M}}(\Omega)^* = \{\widehat{g} : g \in C(\Omega)\}$ , the equation (1) means that the limiting log-moment generating function associated with  $(\nu_\alpha, t_\alpha)$  exists and coincides with the map

$$\widetilde{\mathcal{M}}(\Omega)^* \ni \widehat{g} \mapsto P^\tau(f + g) - P^\tau(f)$$

(the net  $(\nu_\alpha)$  is thought of as a net of measures on  $\widetilde{\mathcal{M}}(\Omega)$ ). In all cases, the exponential tightness holds: This is obvious when  $X = \widetilde{\mathcal{M}}(\Omega)$  (resp.  $X = \mathcal{M}(\Omega)$ ) since  $(\nu_\alpha)$  is supported by the compact set  $\mathcal{M}(\Omega)$ ; in particular, (4) holds for all  $\lambda \in X^*$ ; it is known that in such a situation the large deviation principle in  $\widetilde{\mathcal{M}}(\Omega)$  with rate function  $I$  is equivalent to the large deviation principle in  $\mathcal{M}(\Omega)$  with rate function  $I|_{\mathcal{M}(\Omega)}$ ; furthermore,  $I$  takes the value  $+\infty$  on  $\widetilde{\mathcal{M}}(\Omega) \setminus \mathcal{M}(\Omega)$  ([6], Lemma 4.1.5). When  $X = \mathbb{R}^n$ , the exponential tightness follows from the finiteness of the limiting log-moment generating function on  $\mathbb{R}^n$ , which will always be the case with the nets we shall consider. In the above setting, a large deviation principle in  $\mathcal{M}(\Omega)$  (resp.  $\mathbb{R}^n$ ) is commonly referred as level-2 (resp. level-1).

The result of convex analysis recalled in the last sentence of §2.2 will be applied in particular with  $I^*$  the limiting log-moment generating function associated with  $((\widehat{f}_1, \dots, \widehat{f}_n)[\nu_\alpha]), t_\alpha)$ , where  $(\nu_\alpha, t_\alpha)$  fulfils (1) and  $f_1, \dots, f_n$  are suitable elements of  $C(\Omega)$ ; the function  $I$  will be the rate function governing the large deviation principle.

#### 2.4. Linking large deviations with thermodynamic formalism by convex analysis.

Given  $f \in C(\Omega)$ , the relation (1) is a crucial equality that not only makes the bridge between the large deviation theory and thermodynamic formalism by relating the limiting log-moment generating function  $L_f$  associated with  $(\nu_\alpha, t_\alpha)$  to the pressure function  $P^\tau$ , but when furthermore  $(\nu_\alpha)$  satisfies a large deviation principle in  $\mathcal{M}(\Omega)$  with powers  $(t_\alpha)$  and convex rate function  $I_f$ , it allows to express most basic ingredients of thermodynamic formalism in terms of  $I_f$ ; in particular, (D) can be formulated in terms of a well-known sufficient condition on a convex rate function to get the large deviation principle (*cf.* [3], Theorem 2.1, Property 5.1 and Theorem 5.2). Indeed, in this case,  $I_f = L_{f|_{\mathcal{M}(\Omega)}}^*$  and the

net  $(\nu_\alpha)$  satisfies a large deviation principle in  $\widetilde{\mathcal{M}}(\Omega)$  with powers  $(t_\alpha)$  and rate function  $\widetilde{I}_f = L_f^*$  (cf. §2.3); since

$$\forall \mu \in \widetilde{\mathcal{M}}(\Omega), \quad L_f^*(\mu) = \sup_{\widehat{g} \in \widetilde{\mathcal{M}}(\Omega)^*} \{\widehat{g}(\mu) - L_f(\widehat{g})\} = \sup_{g \in C(\Omega)} \{\mu(g) - Q_f(g)\} = Q_f^*(\mu),$$

where  $Q_f$  is the map defined on  $C(\Omega)$  by

$$\forall g \in C(\Omega), \quad Q_f(g) = P^\tau(f + g) - P^\tau(f),$$

we get (cf. Lemma 8),

$$\forall \mu \in \widetilde{\mathcal{M}}(\Omega), \quad \widetilde{I}_f(\mu) = \begin{cases} P^\tau(f) - h^\tau(\mu) - \mu(f) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \widetilde{\mathcal{M}}(\Omega) \setminus \mathcal{M}^\tau(\Omega). \end{cases}$$

Therefore,  $\mathcal{M}^\tau(\Omega)$  is the effective domain of  $I_f$ ; the entropy map  $h^\tau$  coincides with  $I_f|_{\mathcal{M}^\tau(\Omega)}$  modulo an affine function; given a measure  $\mu \in \mathcal{M}(\Omega)$ ,  $\mu$  is an equilibrium state for  $f$  if and only if  $I_f(\mu) = 0$ ,  $\mu$  is ergodic if and only if  $\mu$  is the unique zero of  $I_f$  for some  $f \in C(\Omega)$ ; for every net  $(\mu_i)$  of ergodic measures,  $\lim_i(\mu_i, h^\tau(\mu_i)) = (\mu, h^\tau(\mu))$  if and only if  $\lim_i(\mu_i, I_f(\mu_i)) = (\mu, I_f(\mu))$ . It follows from the above correspondences that (D) is equivalent to Property 5.1 of [3], which is nothing but a particular case of the condition appearing in Baldi's theorem in large deviation theory (cf. [3], Theorem 2.1 and the proof of Theorem 3.3).

It is worth noticing that (1) is also necessary in order that  $(\nu_\alpha)$  satisfy the large deviation principle in  $\mathcal{M}(\Omega)$  (resp.  $\widetilde{\mathcal{M}}(\Omega)$ ) with powers  $(t_\alpha)$  and the above convex rate function  $I_f$  (resp.  $\widetilde{I}_f$ ); this is a consequence of Varadhan's theorem; indeed, in this case, since (4) holds for all  $\lambda \in X^*$  with  $X = \widetilde{\mathcal{M}}(\Omega)$ , the limiting log-moment generating function  $L_f$  associated with  $(\nu_\alpha, t_\alpha)$  exists as a convex lower semi-continuous function on  $X$  and fulfils  $L_f = \widetilde{I}_f^*$  by (5); since  $\widetilde{I}_f = Q_f^*$  and  $Q_f$  is continuous we have  $Q_f^{**} = Q_f$  hence  $L_f(\widehat{g}) = Q_f(g)$  for all  $g \in C(\Omega)$ , which is exactly (1).

### 3. RESULTS

As a natural candidate for  $(\nu_\alpha, t_\alpha)$  as in (1), for each  $f \in C(\Omega)$  we introduce a basic net  $(\nu_{f,\alpha}^\tau, t_\alpha^\tau)$  canonically associated to the system  $(\Omega, \tau)$ : Let  $\wp$  denote the product set  $]0, +\infty[ \times \mathbb{N}^{l^{]0, +\infty[}}$  pointwise directed, where  $]0, +\infty[$  (resp.  $\mathbb{N}, \mathbb{N}^l$ ) is endowed with the inverse of the natural order on  $\mathbb{R}$  (resp. natural order, lexicographic order), *i.e.*  $(\varepsilon, u) \in \wp$  is less than or equal  $(\varepsilon', u') \in \wp$  if  $\varepsilon \geq \varepsilon'$  and  $u(\delta)$  is lexicographically less than or equal  $u'(\delta)$  for all  $\delta \in ]0, +\infty[$  (cf. [11]). For each  $\alpha = (\varepsilon, u) \in \wp$  we put

$$t_\alpha^\tau = \frac{1}{|\Lambda(u(\varepsilon))|}$$

and

$$\forall f \in C(\Omega), \quad \nu_{f,\alpha}^\tau = \sum_{\xi \in \Omega_{\varepsilon, u(\varepsilon)}} \frac{e^{\sum_{x \in \Lambda(u(\varepsilon))} f(\tau^x \xi)}}{\sum_{\xi' \in \Omega_{\varepsilon, u(\varepsilon)}} e^{\sum_{x \in \Lambda(u(\varepsilon))} f(\tau^x \xi')}} \delta_{\frac{1}{|\Lambda(u(\varepsilon))|} \sum_{x \in \Lambda(u(\varepsilon))} \delta_{\tau^x(\xi)}}.$$

The first equality in (3) implies that  $(\nu_{f,\alpha}^\tau, t_\alpha^\tau)$  fulfils (1) (Lemma 2); this is obvious when  $\tau$  is expansive, in which case the above net is in fact a sequence indexed by elements of  $\mathbb{N}^l$ .

Here is the main result.

**Theorem 1.** *Let  $f_0 \in C(\Omega)$ , let  $E$  be a set generating a  $\sigma$ -compact vector space  $W$  dense in  $C(\Omega)$  and let  $\Sigma$  be a nonempty subset of  $W$ .*

a) *The following statements are equivalent:*

- (i)  $\{\mu \in \mathcal{E}^\tau(\Omega) : h^\tau(\mu) > r\}$  is dense in  $\{\mu \in \mathcal{M}^\tau(\Omega) : h^\tau(\mu) > r\}$  for all  $r \in \mathbb{R}$ ;
- (D)  $\equiv$  (ii) *The graph of  $h^\tau|_{\mathcal{E}^\tau(\Omega)}$  is dense in the graph of  $h^\tau$ .*
- (iii) *For each  $n \in \mathbb{N}$  there exists a set  $D_n$  dense in  $C(\Omega)^n$  such that for each  $(g_1, \dots, g_n) \in D_n$  and for each  $\varepsilon > 0$  there is  $g \in C(\Omega) \setminus \text{span}\{g_1, \dots, g_n\}$  with  $\|g\| < \varepsilon$  such that  $g + \sum_{k=1}^n t_k g_k$  has a unique equilibrium state for all  $(t_1, \dots, t_n) \in \mathbb{R}^n$ .*
- (iv) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that  $f + g$  has a unique equilibrium state for all  $(f, g) \in \Sigma \times V \setminus \{0\}$ .*
- (v) *The net  $(\nu_{f_0,\alpha}^\tau)$  satisfies a large deviation principle in  $\mathcal{M}(\Omega)$  with powers  $(t_\alpha^\tau)$  and a convex rate function  $I$  such that the graph of  $I|_{\mathcal{E}^\tau(\Omega)}$  is dense in the graph of  $I|_{\mathcal{M}^\tau(\Omega)}$ .*
- (vi) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that for each  $(f, g) \in \Sigma \times V \setminus \{0\}$  the net  $(\nu_{f+g,\alpha}^\tau)$  satisfies a large deviation principle in  $\mathcal{M}(\Omega)$  with powers  $(t_\alpha^\tau)$  and a convex rate function vanishing at a unique point.*
- (vii) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the net  $((\widehat{f}_1, \dots, \widehat{f}_n)[\nu_{f+g,\alpha}^\tau])$  satisfies a large deviation principle in  $\mathbb{R}^n$  with powers  $(t_\alpha^\tau)$  and an essentially strictly convex rate function for all  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$ .*
- (viii) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the map*

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \left\{ \sum_{k=1}^n t_k x_k - P^\tau \left( f + g + \sum_{k=1}^n t_k f_k \right) \right\}$$

*is essentially strictly convex for all  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$ .*

- (ix) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the map*

$$\mathbb{R}^n \ni x \mapsto \inf \{ -\mu(f + g) - h^\tau(\mu) : \mu \in M^\tau(\Omega), (\mu(f_1), \dots, \mu(f_n)) = x \}$$

*is essentially strictly convex for all  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$ .*

Furthermore:

- 1) *If an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  fulfils one of the conditions (iv), (vi), (vii), (viii), (ix), then  $V$  fulfils all of them.*
- 2) *The rate function in (v) (resp. (vi)) is*

$$\mathcal{M}(\Omega) \ni \mu \mapsto \begin{cases} P^\tau(f_0) - h^\tau(\mu) - \mu(f_0) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \setminus \mathcal{M}^\tau(\Omega) \end{cases}$$



$$\left( \text{resp. } \mathcal{M}(\Omega) \ni \mu \mapsto \begin{cases} P^\tau(f+g) - h^\tau(\mu) - \mu(f+g) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \setminus \mathcal{M}^\tau(\Omega) \end{cases} \right),$$

and the rate function  $I_{f+g, (f_1, \dots, f_n)}$  in (vii) fulfils for each  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$I_{f+g, (f_1, \dots, f_n)}(x_1, \dots, x_n) - P^\tau(f+g) = \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \left\{ \sum_{k=1}^n t_k x_k - P^\tau \left( f+g + \sum_{k=1}^n t_k f_k \right) \right\} \\ = \inf \{ -\mu(f+g) - h^\tau(\mu) : \mu \in M^\tau(\Omega), (\mu(f_1), \dots, \mu(f_n)) = (x_1, \dots, x_n) \}.$$

3) In (iii) one can take  $D_n = V^n$  with  $V$  as in (iv) (resp. (vi), (vii), (viii), (ix)).

b) Part a) holds verbatim with any one of the following changes:

1) Replacing in (v) the net  $(\nu_{f_0, \alpha}^\tau)$  and  $(t_\alpha^\tau)$  respectively by any net  $(\nu_\alpha)$  of Borel probability measures on  $\mathcal{M}(\Omega)$  and any net  $(t_\alpha)$  of positive real numbers converging to zero fulfilling for each  $h \in C(\Omega)$ ,

$$\lim_{\alpha} t_\alpha \log \int_{\mathcal{M}(\Omega)} e^{t_\alpha^{-1} \int_{\Omega} h(\omega) \mu(d\omega)} \nu_\alpha(d\mu) = P^\tau(f_0 + h) - P^\tau(f_0);$$

2) Replacing in (vi) the net  $(\nu_{f+g, \alpha}^\tau)$  and  $(t_\alpha^\tau)$  respectively by any net  $(\nu_\alpha)$  of Borel probability measures on  $\mathcal{M}(\Omega)$  and any net  $(t_\alpha)$  of positive real numbers converging to zero fulfilling for each  $h \in C(\Omega)$ ,

$$\lim_{\alpha} t_\alpha \log \int_{\mathcal{M}(\Omega)} e^{t_\alpha^{-1} \int_{\Omega} h(\omega) \mu(d\omega)} \nu_\alpha(d\mu) = P^\tau(f+g+h) - P^\tau(f+g);$$

3) Replacing in (vii) the net  $((\widehat{f_1}, \dots, \widehat{f_n})[\nu_{f+g, \alpha}^\tau])$  and  $(t_\alpha^\tau)$  respectively by any net  $(\mu_\alpha)$  of Borel probability measures on  $\mathbb{R}^n$  and any net  $(t_\alpha)$  of positive real numbers converging to zero fulfilling for each  $(t_1, \dots, t_n) \in \mathbb{R}^n$ ,

$$\lim_{\alpha} t_\alpha \log \int_{\mathbb{R}^n} e^{t_\alpha^{-1} \sum_{k=1}^n t_k x_k} \mu_\alpha(d(x_1, \dots, x_n)) = P^\tau(f+g + \sum_{k=1}^n t_k f_k) - P^\tau(f+g).$$

4) Requiring furthermore that  $V \cap W = \{0\}$  in (iv) (resp. (vi), (vii), (viii), (ix)).

c) If each element of  $\Sigma$  has a unique equilibrium state, then part a) holds verbatim replacing  $V \setminus \{0\}$  by  $V$ . If one of the conditions (iv), (vi), (vii), (viii), (ix) holds replacing  $V \setminus \{0\}$  by  $V$ , then all these conditions hold replacing  $V \setminus \{0\}$  by  $V$  (in particular, each element of  $\Sigma$  has a unique equilibrium state) and part b) remains true with this change.

When  $W$  contains an element admitting several equilibrium states, the proof of Theorem 1 reveals that  $V$  is obtained as a proper subspace of a direct sum  $W \oplus \widetilde{W}$ , where  $\widetilde{W}$  is a  $\sigma$ -compact infinite dimensional vector subspace of  $C(\Omega)$  such that  $f+h$  has a unique equilibrium state for all  $(f, h) \in W \times (\widetilde{W} \setminus \{0\})$ ; the space  $\widetilde{W}$  is given by Lemma 12 and  $V$  is obtained by means of a Schauder basis of  $C(\Omega)$  included in  $W$ ; the existence of such a space  $\widetilde{W}$  can in turn be recovered from Theorem 1: The condition (iv) implies  $V \cap W = \{0\}$  so that  $\widetilde{W}$  can be taken as any  $\sigma$ -compact infinite dimensional vector subspace of  $V$ . When each

element of  $W$  has a unique equilibrium state, one considers the space  $W' = W \oplus \text{span}(\{f\})$ , where  $f \in C(\Omega)$  has several equilibrium states (such an element exists by Theorem 3.4 of [9], cf. Lemma 11) and the preceding case applies with  $W'$  in place of  $W$ . Note that in all cases, the space  $\widetilde{W}$  so obtained may furthermore be chosen as to be dense in  $V$ , and thus dense in  $C(\Omega)$ , which is an extra property that is not given by Lemma 12 neither by Theorem 3 in Appendix A on which Lemma 12 is based.

The following theorem specifies the nature of  $V$  and explains the use of Schauder bases; it establishes a method that permits us from any  $\sigma$ -compact space  $V$  as above to get a new one  $V_1$  linearly independent from  $V$  and such that  $V \oplus V_1$  fulfils the same properties as  $V$ ; by iterating this process, we obtain an infinite direct sum  $\bigoplus_{n=0}^{\infty} V_n$  with  $V_0 = V$ , whose existence furnishes a new criterion for the validity of (D); furthermore, starting with  $\bigoplus_{n=1}^{\infty} V_n$  and using only subspaces of this sum, the above method allows us to build another direct sum  $\bigoplus_{n=1}^{\infty} \widetilde{V}_n$  linearly independent from  $\bigoplus_{n=1}^{\infty} V_n$  and fulfilling the same properties as  $\bigoplus_{n=1}^{\infty} V_n$ .

**Theorem 2.** *Let  $E$ ,  $\Sigma$  and  $W$  be as in Theorem 1. Let  $V$  be an infinite dimensional vector space dense in  $C(\Omega)$  fulfilling one of the conditions (iv), (vi), (vii), (viii), (ix) of Theorem 1 with  $E$ ,  $\Sigma$  and  $W$ . Let  $V_0$  be a  $\sigma$ -compact vector space dense in  $V$ .*

- a) *There exists an infinite direct sum  $\bigoplus_{n=1}^{\infty} V_n$ , where each  $V_n$  is a  $\sigma$ -compact infinite dimensional vector space dense in  $C(\Omega)$  and linearly independent from  $V_0$ , such that for each nonempty subset  $\mathcal{N}$  of  $\mathbb{N} \cup \{0\}$  the direct sum  $\bigoplus_{n \in \mathcal{N}} V_n$  fulfils all the conditions (iv), (vi), (vii), (viii), (ix) of Theorem 1 applied with  $W + \bigoplus_{n \in \mathbb{N} \cup \{0\} \setminus \mathcal{N}} V_n$  in place of  $W$  (putting  $\bigoplus_{n \in \emptyset} V_n = \{0\}$  when  $\mathcal{N} = \mathbb{N} \cup \{0\}$ ).*
- b) *An infinite direct sum as in part a) may be obtained by recurrence in the following way: For each  $n \in \mathbb{N} \cup \{0\}$ , given  $V_0, \dots, V_n$ , we consider a  $\sigma$ -compact infinite dimensional vector space  $\widetilde{W}_n$  such that  $f + g$  has a unique equilibrium state for all  $(f, g) \in W + \bigoplus_{j=0}^n V_j \times (\widetilde{W}_n \setminus \{0\})$  and we put*

$$V_{n+1} = \text{span}(\{f_{n,k} + h_{n,k} : k \in \mathbb{N}\}),$$

*where  $(f_{n,k})$  is a Schauder basis of  $C(\Omega)$  included in  $W + \bigoplus_{j=0}^n V_j$  and  $\{h_{n,k} : k \in \mathbb{N}\}$  a linearly independent subset of  $\widetilde{W}_n$  fulfilling*

$$\sum_{k=1}^{+\infty} \left( \sup_{f \in C(\Omega), \|f\| \leq 1} |\lambda_{n,k}(f)| \right) \|h_{n,k}\| < 1, \quad (6)$$

*where  $(\lambda_{n,k}(f))$  denotes the coordinates of  $f$  in the basis  $(f_{n,k})$ . Furthermore, if  $V_0 \cap W = \{0\}$  then the above assertion holds verbatim replacing  $W + V_0$  by  $W$  and taking  $\widetilde{W}_0 = V_0$ ; this is the case in particular when  $W$  contains an element admitting several equilibrium states.*

- c) *Let  $\bigoplus_{n=1}^{\infty} V_n$  be as in part a), let  $(m_n)$  be a strictly increasing sequence in  $\mathbb{N}$  and for each  $n \in \mathbb{N}$  let  $V_n$  be a  $\sigma$ -compact infinite dimensional vector subspace of  $\bigoplus_{j=m_n}^{j=m_{n+1}} V_j$ .*

Put  $\tilde{V}_0 = V_0$  and

$$\forall n \in \mathbb{N} \cup \{0\}, \quad \tilde{V}_{n+1} = \text{span}(\{f_{n,k} + h_{n,k} : k \in \mathbb{N}\}),$$

where  $(f_{n,k})$  is a Schauder basis of  $C(\Omega)$  included in  $W + \bigoplus_{j=0}^n \tilde{V}_j$  and  $(h_{n,k})$  a linearly independent subset of  $\mathcal{V}_{n+1}$  fulfilling (6). Then, the direct sum  $\bigoplus_{n=1}^{\infty} \tilde{V}_n$  is linearly independent from  $\bigoplus_{n=1}^{\infty} V_n$  and fulfils the same properties as  $\bigoplus_{n=1}^{\infty} V_n$  stated in part a). If  $V_0 \cap W = \{0\}$ , then the above assertion holds verbatim replacing  $W + \tilde{V}_0$  (resp.  $\mathcal{V}_1$ ) by  $W$  (resp.  $V_0$ ).

The conclusions of part c) of Theorem 1 with  $\Sigma = \{0\}$  concerning the level-1 large deviations and the convex functions related with the associated rate functions (namely, conditions (vii), (viii) and (ix)) can be substantially improved as shows the following corollary.

**Corollary 1.** *The following statements are equivalent:*

- (i) *There exists a unique measure of maximal entropy and the graph of  $h^\tau|_{\mathcal{E}^\tau(\Omega)}$  is dense in the graph of  $h^\tau$ .*
- (ii) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that each element of  $V$  has a unique equilibrium state;*
- (iii) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the map*

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \left\{ \sum_{k=1}^n t_k x_k - P^\tau \left( \sum_{k=1}^n t_k f_k \right) \right\}$$

*is essentially strictly convex for all  $((f_1, \dots, f_n), n) \in V^n \times \mathbb{N}$ .*

- (iv) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the map*

$$\mathbb{R}^n \ni x \mapsto \inf \{ -h^\tau(\mu) : \mu \in M^\tau(\Omega), (\mu(f_1), \dots, \mu(f_n)) = x \}$$

*is essentially strictly convex for all  $((f_1, \dots, f_n), n) \in V^n \times \mathbb{N}$ .*

- (v) *There exists an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  such that the net  $((\hat{f}_1, \dots, \hat{f}_n)[\nu_{0,\alpha}^\tau])$  satisfies a large deviation principle in  $\mathbb{R}^n$  with powers  $(t_\alpha^\tau)$  and an essentially strictly convex rate function for all  $((f_1, \dots, f_n), n) \in V^n \times \mathbb{N}$ .*

Furthermore:

- 1) *If an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  fulfils one of the conditions (ii), (iii), (iv), (v), then for each  $\sigma$ -compact vector space  $V_0$  dense in  $V$  there exists an infinite direct sum  $\bigoplus_{n=1}^{\infty} V_n$ , where each  $V_n$  is a  $\sigma$ -compact infinite dimensional vector space dense in  $C(\Omega)$  and linearly independent from  $V_0$ , such that  $\bigoplus_{n=0}^{\infty} V_n$  fulfils all the conditions (ii), (iii), (iv), (v).*
- 2) *The above equivalences hold verbatim replacing in (v) the net  $((\hat{f}_1, \dots, \hat{f}_n)[\nu_{0,\alpha}^\tau])$  and  $(t_\alpha^\tau)$  respectively by any net  $(\mu_\alpha)$  of Borel probability measures on  $\mathbb{R}^n$  and any net  $(t_\alpha)$  of positive real numbers converging to zero fulfilling for some (arbitrary and independent of  $n$ )  $f \in V$  and for each  $(t_1, \dots, t_n) \in \mathbb{R}^n$ ,*

$$\lim_{\alpha} t_\alpha \log \int_{\mathbb{R}^n} e^{t_\alpha^{-1} \sum_{k=1}^n t_k x_k} \mu_\alpha(d(x_1, \dots, x_n)) = P^\tau \left( f + \sum_{k=1}^n t_k f_k \right) - P^\tau(f).$$

In this case, the rate function  $I_{f,(f_1,\dots,f_n)}$  governing the large deviation principle of the above net (and in particular the net  $((\widehat{f}_1, \dots, \widehat{f}_n)[\nu_{f,\alpha}^\tau])$  of (v) with  $f = 0$ ) fulfils for each  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$\begin{aligned} I_{f,(f_1,\dots,f_n)}(x_1, \dots, x_n) - P^\tau(f) &= \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \left\{ \sum_{k=1}^n t_k x_k - P^\tau \left( f + \sum_{k=1}^n t_k f_k \right) \right\} \\ &= \inf \{ -\mu(f) - h^\tau(\mu) : \mu \in M^\tau(\Omega), (\mu(f_1), \dots, \mu(f_n)) = (x_1, \dots, x_n) \}. \end{aligned}$$

*Remark 1.* The statements (i), (ii) and (iii) of Theorem 1 depend neither on  $\Sigma$  nor on  $E$  nor on  $f_0$ . Therefore, Theorem 1 holds verbatim replacing  $\Sigma$  by any non-empty subset of  $W$  in (iv) (resp. (vi), (vii), (viii), (ix)); similarly, one can replace  $E$  by any set generating  $W$  in (vii) (resp. (viii), (ix)); in other words, the sets  $\Sigma$  and  $E$  may be different in each condition where they appear. The condition (v) holds for some  $f_0 \in C(\Omega)$  if and only if (v) holds for all  $f_0 \in C(\Omega)$ .

*Remark 2.* The space  $\widetilde{W}_n$  in part b) of Theorem 2 may be obtained by Lemma 12 as well as by applying Theorem 1 with  $W + \bigoplus_{j=0}^n V_j$  in place of  $W$ . In any case, the proof reveals that given a linearly independent subset  $\{h_k : k \in \mathbb{N}\}$  of  $\widetilde{W}_n$ , any set  $\{f_k : k \in \mathbb{N}\} \subset W + \bigoplus_{j=0}^n V_j$  such that  $\text{span}(\{f_k + h_k : k \in \mathbb{N}\})$  is dense in  $C(\Omega)$  would do (cf. first assertion of Lemma 14). It turns out that the concept of Schauder basis captures more than the properties needed: indeed, when  $(f_k)$  is a Schauder basis of  $C(\Omega)$ , there is a sequence  $(\varepsilon_k)$  of positive real numbers converging to zero such that the sequence  $(f_k + h'_k)$  is a Schauder basis for every sequence  $(h'_k)$  satisfying

$$\forall k \in \mathbb{N}, \quad \|h'_k\| \leq \varepsilon_k \quad (7)$$

(cf. §2.1); therefore, when  $\|h_k\| \leq \varepsilon_k$  for all  $k \in \mathbb{N}$ , this allows us to choose not only  $(h'_k) = (h_k)$ , but any linearly independent set  $\{h'_k : k \in \mathbb{N}\} \subset \widetilde{W}_n$  fulfilling (7); note that the linear independence of  $\{h'_k : k \in \mathbb{N}\}$  is necessary since  $\{f_k : k \in \mathbb{N}\}$  is linearly independent (cf. second assertion of Lemma 14). The same observation holds regarding the proof of the implication (i)  $\Rightarrow$  (iv<sub>4</sub>) of Theorem 1.

*Remark 3.* The condition (ii) (resp. (iii), (iv), (v) with the changes 2)) of Corollary 1 with  $V$   $\sigma$ -compact is equivalent to the condition (iv) (resp. (viii), (ix), (vii) with the changes b)3)) of Theorem 1 with  $\Sigma = W = V = E$  and replacing  $V \setminus \{0\}$  by  $V$ . Therefore (by the properties a)1) and c) of Theorem 1) given  $V$  as in Corollary 1, all the conditions in part a) of Theorem 1 (as well as those obtained with the changes b)1), b)2), b)3)) hold with  $W$  any  $\sigma$ -compact vector space dense in  $V$  and replacing  $V \setminus \{0\}$  by  $V$  in (iv), (vi), (vii), (viii), (ix).

It is worth putting into perspective the achievements of Theorem 1 as regards their proofs and respective relevance: The equivalences of Theorem 1 involving (D) (except (D)  $\Leftrightarrow$  (i), which is only technical and given by Lemma 1) can be classified in three categories depending on the terms in which (D) is formulated:

- variational form of the pressure in an infinite dimensional setting in (iii), (iv);
- topological form of the pressure in (v), (vi), (vii), b)1), b)2), b)3) (i.e. involving the large deviations);
- variational form of the pressure or entropy in a finite dimensional setting in (viii), (ix).

Once the equivalences of the first above class are obtained, the ones of the second class are essentially based on the variational principle (thanks to Lemma 2); deriving those of the last class from the preceding ones is done by means of convex analysis, in finite as well as infinite dimensional setting. So, regardless the large deviation aspect, the main achievement of Theorem 1 is certainly the equivalences  $(D) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (viii) \Leftrightarrow (ix)$  and especially  $(D) \Leftrightarrow (iv)$ , the proof of which relies on a variant of two results of [9] (namely, Theorem 3 and Theorem 4 of Appendix A) given by Lemma 12 and Lemma 13; Schauder basis are used for the proof of the implication  $(D) \Rightarrow (iv)$ .

From the large deviation point of view, the fact that (D) implies the level-2 large deviation results (v) with the changes b)1), (vi) with the changes b)2) and the form of the rate function given in a)2) is well-known ([3], Theorem 5.2); the statements (v) and (vi) follow also from Theorem 5.2 of [3] once it is known that  $(\nu_{f,\alpha}^\tau, t_\alpha^\tau)$  fulfils (1), which is given by Lemma 2. The novelties are the converse implications, which follow easily from  $(D) \Leftrightarrow (iv)$  and the form of the level-2 rate function (Lemma 8).

The key ingredient that allows to prove the equivalence between (D) and the level-1 large deviations results (vii) (resp. (vii) with the changes b)3)) and especially the essential strict convexity of the rate function is the peculiar form of the limiting log-moment generating function (cf. Lemma 5 and Lemma 7); since the proof does not depend on the nature of the nets satisfying these large deviation principles, once the existence of such a net is established (which is given by (vii)) one also obtains  $(D) \Leftrightarrow (viii)$ ; once the latter is known, the equivalence  $(D) \Leftrightarrow (ix)$  relies on Lemma 9 and Lemma 10.

**Example 1.** Let  $(\Omega, \tau)$  be the system given by the iteration of a rational map  $T$  of degree at least two ([1]). More precisely,  $\Omega$  is the Julia set of  $T$  endowed with the induced chordal metric, and the action  $\tau$  is defined by

$$\mathbb{N} \cup \{0\} \ni n \mapsto \tau(n) = (T|_\Omega)^n;$$

such a system has a unique measure of maximal entropy ([13]). We assume furthermore that  $T$  fulfils a weak form of hyperbolicity, the so-called Topological Collet-Eckman (TCE) condition: There exists  $\lambda > 1$  such that every periodic point  $p \in \Omega$  with period  $n$  satisfies

$$|(T^n)'(p)| \geq \lambda^n.$$

(see Main Theorem of [15] for other equivalent definitions). Let  $H$  denote the space of Hölder continuous functions on  $\Omega$ ; since every element of  $H$  has a unique equilibrium state (Theorem A of [2]) we obtain the following:

- The conditions (ii), (iii), (iv), (v) of Corollary 1 hold with  $V = H$ . All the conditions in part a) (and thus also those given by the changes b)1), b)2), b)3)) of Theorem 1 hold; consequently, the hypotheses (and thus the properties a), b), c)) of Theorem 2 are

fulfilled. Furthermore, when  $W$  is a  $\sigma$ -compact vector space dense in  $H$ , the conditions (iv), (vi), (vii), (viii), (ix) of Theorem 1 hold with  $V = H$  and replacing  $V \setminus \{0\}$  by  $V$  (cf. Remark 3).

- The foregoing allows us to generalize and strengthen all the level-2 as well as level-1 large deviation results of [2]; as regards the level-2, the strengthening consists in the property of the rate function given by (v) and b)1); with respect to level-1, the improvement is given by the essential strict convexity of the rate function and the fact that this holds in any finite dimension; in both cases, the generalization is done by considering other nets (for instance,  $(\nu_{f,\alpha}^\tau, t_\alpha^\tau)$ ) as well as allowing the parameter function  $f$  (as in (1)) to be an arbitrary element of  $C(\Omega)$ . More specifically, the connection with [2] works as follows:
  - The level-2 large deviation results of [2] (namely, Theorem B) follow from Theorem A of [2] together with Theorem 5.2 of [3] in place of Theorem C of [2]; alternatively, in place of Theorem 5.2 of [3] one can use the condition (vi) and b)2) of Theorem 1 with  $\Sigma = \{0\}$  (the equality in b)2) for the nets considered in [2] is proved in Lemmas 4.2, 4.3 and 4.4 of that paper). Furthermore, the statements (v), (vi) and (vii) of Theorem 1 furnish large deviations results that have not been considered in [2]. It is easy to derive them using Theorem A and Theorem C of [2] and Lemma 2 when  $f_0 \in H$  and  $\Sigma \subset H$ ; however, the statement (v) when  $f_0 \notin H$  and the statements (vi), (vii) when  $\Sigma \not\subset H$  required (D) and thus cannot be obtained directly from [2]; indeed, one needs to know that Theorem A of [2] implies (D) (in other words, (ii)  $\Rightarrow$  (i) of Corollary 1).
  - The level-1 large deviation results of [2] (namely, Corollary 1.1 of that paper) are strengthened by the statements (vii) together with b)3) of Theorem 1 (alternatively, by (v) and the assertion 2) of Corollary 1) by giving a  $n$ -dimensional version for all  $n \in \mathbb{N}$  and establishing the essential strict convexity of the rate function; in particular, the last assertion of Corollary 1.1 of [2] is a direct consequence of this fact. The same improvements apply to Theorem 3.5 of [3] concerning the level-1 large deviation result for hyperbolic rational maps (*i.e.* expanding on  $\Omega$ ): indeed, the hyperbolicity implies that both the parameter function (*i.e.*  $-t \log |T'|$ ) as well as the function by which the net is pushed forward (*i.e.*  $\log |T'|$ ) belong to  $H$ .

*Remark 4.* The implication (iv)  $\Rightarrow$  (v) with the changes b)1) of Theorem 1 strengthens the large deviation results of Theorem C of [2] (as well as their generalization given by Remark B.2) by weakening the general hypothesis, *i.e.* allowing  $f_0$  to have several equilibrium states, and keeping the conclusions (namely, the first and the last assertions) unchanged.

#### 4. PROOFS

**Lemma 1.** *Let  $J$  and  $L$  be directed sets, let  $s$  be a real-valued function on  $J \times L$ , let  $\wp$  denote the set  $J \times L^J$  pointwise directed, let  $(s_i)_{i \in \wp}$  be the net in  $\mathbb{R}$  defined by putting  $s_i = s(j, u(j))$*

for all  $i = (j, u) \in \wp$ . For each  $r \in \mathbb{R}$  we have

$$\limsup_i s_i \leq r \iff \limsup_j \limsup_l s(j, l) \leq r;$$

in particular,

$$\lim_i s_i = r \iff \liminf_j \liminf_l s(j, l) = \limsup_j \limsup_l s(j, l) = r.$$

*Proof.* Let  $\delta > 0$ . First assume that  $\limsup_i s_i \leq r$ . There exists  $j_0 \in J$  and  $u_0 \in L^J$  such that  $s(j, u(j)) < r + \delta$  for all  $(j, u)$  greater than or equal  $(j_0, u_0)$ . Suppose that  $\limsup_j \limsup_l s(j, l) > r + \delta$ . There exists  $(j_1, l_1)$  in  $J \times L$  with  $j_1$  (resp.  $l_1$ ) greater than or equal  $j_0$  (resp.  $u_0(j_1)$ ) such that  $s(j_1, l_1) > r + \delta$ . Putting  $u_1(j_1) = l_1$  and  $u_1(j) = u_0(j)$  for all  $j \in J \setminus \{j_1\}$ , we get an element  $(j_1, u_1) \in \wp$  greater than or equal  $(j_0, u_0)$  fulfilling  $s(j_1, u_1(j_1)) > r + \delta$ , which gives the contradiction. Therefore, we have  $\limsup_j \limsup_l s(j, l) \leq r + \delta$  hence  $\limsup_j \limsup_l s(j, l) \leq r$  since  $\delta$  is arbitrary.

Assume now that  $\limsup_j \limsup_l s(j, l) \leq r$ . There exists  $j_0 \in J$  and for each  $j \in J$  greater than or equal  $j_0$  there exists  $u_0(j) \in L$  such that  $s(j, l) < r + \delta$  for all  $j$  and  $l$  greater than or equal  $j_0$  and  $u_0(j)$ , respectively. Putting  $u_0(j) = u_0(j_0)$  for all  $j$  lesser than  $j_0$ , we get an element  $(j_0, u_0) \in \wp$  such that  $s(j, u(j)) < r + \delta$  for all  $(j, u) \in \wp$  greater than or equal  $(j_0, u_0)$ ; therefore,  $\limsup_i s_i \leq r + \delta$  hence  $\limsup_i s_i \leq r$  since  $\delta$  is arbitrary. The first assertion is proved; the second assertion is a direct consequence since  $\liminf_i s_i \geq r$  if and only if  $-\limsup_i -s_i \geq r$  if and only if  $\limsup_i -s_i \leq -r$  if and only if  $\limsup_j \limsup_l -s(j, l) \leq -r$  if and only if  $-\liminf_j \liminf_l s(j, l) \leq -r$  if and only if  $\liminf_j \liminf_l s(j, l) \geq r$  (where the third equivalence follows from the first assertion applied to the net  $(-s_i)$  and the real  $-r$ ).  $\square$

**Lemma 2.** For each  $(f, g) \in C(\Omega)^2$  we have

$$\lim_{\alpha} t_{\alpha} \log \int_{\mathcal{M}(\Omega)} e^{(t_{\alpha}^{\tau})^{-1} \int_{\Omega} g(\omega) \mu(d\omega)} \nu_{f, \alpha}^{\tau}(d\mu) = P^{\tau}(f + g) - P^{\tau}(f).$$

*Proof.* Let  $(f, g) \in C(\Omega)^2$ . For each  $(a, h, \xi) \in \mathbb{N}^l \times C(\Omega) \times \Omega$  we put

$$S_{\Lambda(a)}^{\tau}(h)(\xi) = \sum_{x \in \Lambda(a)} h(\tau^x \xi).$$

We have

$$\begin{aligned}
P^\tau(f+g) - P^\tau(f) &= \\
&\lim_{\varepsilon \rightarrow 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)} - \lim_{\varepsilon \rightarrow 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)} \\
&\leq \lim_{\varepsilon \rightarrow 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)}} \right) \\
&\leq \lim_{\varepsilon \rightarrow 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)}} \right) \\
&\leq \lim_{\varepsilon \rightarrow 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)} \\
&\quad - \lim_{\varepsilon \rightarrow 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)} = P^\tau(f+g) - P^\tau(f),
\end{aligned}$$

hence

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)}} \right) \\
&= \lim_{\varepsilon \rightarrow 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_{\Lambda(a)}^\tau(f)(\xi)}} \right). \quad (8)
\end{aligned}$$

Since for each  $\alpha = (\varepsilon, u) \in \mathcal{P}$ ,

$$t_\alpha^\tau \log \int_{\mathcal{M}(\Omega)} e^{(t_\alpha^\tau)^{-1} \int_\Omega g(\omega) \mu(d\omega)} \nu_{f,\alpha}^\tau(d\mu) = \frac{1}{|\Lambda(u(\varepsilon))|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,u(\varepsilon)}} e^{S_{\Lambda(u(\varepsilon))}^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,u(\varepsilon)}} e^{S_{\Lambda(u(\varepsilon))}^\tau(f)(\xi)}} \right),$$

the conclusion follows from (8) together with Lemma 1 applied with  $J = ]0, +\infty[$  and  $L = \mathbb{N}^l$  with their respective orders (defined before Theorem 1) and  $s$  defined by

$$\forall (\varepsilon, a) \in J \times L, \quad s(\varepsilon, a) = \frac{1}{|\Lambda(a)|} \log \left( \frac{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_a^\tau(f+g)(\xi)}}{\sum_{\xi \in \Omega_{\varepsilon,a}} e^{S_a^\tau(f)(\xi)}} \right).$$

□

**Lemma 3. (Phelps)** *For each  $\mu \in \mathcal{M}^\tau(\Omega)$ ,  $\mu \in \mathcal{E}^\tau(\Omega)$  if and only if  $\mu$  is the unique equilibrium state for some element in  $C(\Omega)$ .*

*Proof.* The fact that each  $\mu \in \mathcal{E}^\tau(\Omega)$  is the unique equilibrium state for some element in  $C(\Omega)$  is exactly Theorem 1 of [14]; the converse follows from the fact that  $h^\tau$  is affine. □



Recall that the directional derivative of  $P^\tau$  at  $f$  in the direction  $g$  is denoted by  $dP^\tau(f; g)$  for all  $(f, g) \in C(\Omega)^2$  (cf. §2.2).

**Lemma 4.** *Let  $f \in C(\Omega)$  and let  $W$  be a vector space dense in  $C(\Omega)$ . If the map  $W \ni g \mapsto dP^\tau(f; g)$  is real-valued and linear, then  $P^\tau$  is Gateaux differentiable at  $f$ .*

*Proof.* Let  $\delta P^\tau(f)$  denote the set of subgradients of  $P^\tau$  at  $f$  (i.e.  $\delta P^\tau(f)$  is the set of equilibrium states for  $f$ , cf. §2.2). For each  $(\mu, g) \in \delta P^\tau(f) \times W$  we have

$$\forall \varepsilon > 0, \quad P^\tau(f + \varepsilon g) - P^\tau(f) \geq \varepsilon \mu(g);$$

dividing by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$  yields  $dP^\tau(f; g) - \mu(g) \geq 0$  hence  $dP^\tau(f; g) = \mu(g)$  by linearity of  $dP^\tau(f; \cdot)|_W$  and  $\mu|_W$ . Since  $W$  is dense in  $C(\Omega)$  it follows that  $\delta P^\tau(f)$  is a singleton, which proves the lemma ([7], Proposition 5.3; cf. §2.2).  $\square$

For each  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$  let  $L_{f, (f_1, \dots, f_n)}$  be the function defined on  $\mathbb{R}^n$  by

$$\forall (t_1, \dots, t_n) \in \mathbb{R}^n, \quad L_{f, (f_1, \dots, f_n)}((t_1, \dots, t_n)) = P^\tau \left( f + \sum_{k=1}^n t_k f_k \right) - P^\tau(f);$$

clearly,  $L_{f, (f_1, \dots, f_n)}$  is real-valued and convex; note that when  $(\nu_\alpha, t_\alpha)$  is a net as in (1),  $L_{f, (f_1, \dots, f_n)}$  is the limiting log-moment generating function associated with the net  $((\widehat{f_1}, \dots, \widehat{f_n})[\nu_\alpha], t_\alpha)$  (§2.3).

**Lemma 5.** *Let  $((f, f_1, \dots, f_n), (t_1, \dots, t_n), n) \in C(\Omega)^{n+1} \times \mathbb{R}^n \times \mathbb{N}$ . If*

$$\forall j \in \{1, \dots, n\}, \quad dP^\tau(f + \sum_{k=1}^n t_k f_k; f_j) = -dP^\tau(f + \sum_{k=1}^n t_k f_k; -f_j),$$

*then  $L_{f, (f_1, \dots, f_n)}$  is differentiable at  $(t_1, \dots, t_n)$ .*

*Proof.* For each  $\varepsilon > 0$  and for each  $j \in \{1, \dots, n\}$  we have

$$\begin{aligned} L_{f, (f_1, \dots, f_n)}((t_1, \dots, t_j + \varepsilon, \dots, t_n)) - L_{f, (f_1, \dots, f_n)}((t_1, \dots, t_n)) \\ = P^\tau \left( f + \sum_{k=1}^n t_k f_k + \varepsilon f_j \right) - P^\tau \left( f + \sum_{k=1}^n t_k f_k \right) \end{aligned}$$

and

$$\begin{aligned} L_{f, (f_1, \dots, f_n)}((t_1, \dots, t_j - \varepsilon, \dots, t_n)) - L_{f, (f_1, \dots, f_n)}((t_1, \dots, t_n)) \\ = P^\tau \left( f + \sum_{k=1}^n t_k f_k - \varepsilon f_j \right) - P^\tau \left( f + \sum_{k=1}^n t_k f_k \right). \end{aligned}$$

Dividing by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$  in both above expressions shows that the  $j^{\text{th}}$ -partial derivative of  $L_{f, (f_1, \dots, f_n)}$  at  $(t_1, \dots, t_n)$  exists and fulfils

$$\frac{\delta^j L_{f, (f_1, \dots, f_n)}}{\delta t_j}(t_1, \dots, t_n) = dP^\tau(f + \sum_{k=1}^n t_k f_k; f_j);$$

furthermore,  $\frac{\delta^j L_{f,(f_1,\dots,f_n)}}{\delta t_j}(t_1, \dots, t_n)$  is finite since the effective domain of  $L_{f,(f_1,\dots,f_n)}$  is  $\mathbb{R}^n$ ; the conclusion follows from Theorem 25.2 of [16].  $\square$

**Lemma 6.** *Let  $f \in C(\Omega)$  and let  $W$  be a vector space dense in  $C(\Omega)$ . If  $L_{f,(g,h)}$  is differentiable at zero for all  $(g,h) \in W^2$ , then  $P^\tau$  is Gateaux differentiable at  $f$ .*

*Proof.* For each  $((g,h), (\lambda, \lambda')) \in W^2 \times \mathbb{R}^2$  and for each  $\varepsilon > 0$  we have

$$L_{f,(g,h)}(\varepsilon(\lambda, \lambda')) = P^\tau(f + \varepsilon(\lambda g + \lambda' h)) - P^\tau(f);$$

dividing by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$  yields

$$dP^\tau(f; \lambda g + \lambda' h) = D_{(0,0)} L_{f,(g,h)}((\lambda, \lambda'))$$

hence  $dP^\tau(f; \cdot)|_W$  is real-valued and linear on  $W$ ; the conclusion follows from Lemma 4.  $\square$

**Lemma 7.** *Let  $f \in C(\Omega)$  and let  $E$  be a set generating a vector space  $W$  dense in  $C(\Omega)$ . If  $L_{f,(f_1,\dots,f_n)}$  is differentiable on  $\mathbb{R}^n$  for all  $((f_1,\dots,f_n), n) \in E^n \times \mathbb{N}$ , then  $P^\tau$  is Gateaux differentiable at  $f + g$  for all  $g \in W$ .*

*Proof.* For each  $((f_1, \dots, f_{\max\{m,m_1,m_2\}}), (t_1, \dots, t_m, u_1, \dots, u_{m_1}, v_1, \dots, v_{m_2}), (m, m_1, m_2)) \in E^{\max\{m,m_1,m_2\}} \times \mathbb{R}^{m+m_1+m_2} \times \mathbb{N}^3$ , for each  $\varepsilon > 0$  and for each  $\varepsilon' > 0$  we have

$$\begin{aligned} & L_{f+\sum_{k=1}^m t_k f_k, (\sum_{k=1}^{m_1} u_k f_k, \sum_{k=1}^{m_2} v_k f_k)}((\varepsilon, \varepsilon')) \\ &= L_{f+\sum_{k=1}^m t_k f_k, (f_1, \dots, f_{m_1}, f_1, \dots, f_{m_2})}((\varepsilon u_1, \dots, \varepsilon u_{m_1}, \varepsilon' v_1, \dots, \varepsilon' v_{m_2})) \\ &= (P^\tau(f + \sum_{k=1}^m t_k f_k + \varepsilon \sum_{k=1}^{m_1} u_k f_k + \varepsilon' \sum_{k=1}^{m_2} v_k f_k) - P^\tau(f)) + (P^\tau(f) - P^\tau(f + \sum_{k=1}^m t_k f_k)) \\ &= L_{f,(f_1, \dots, f_m, f_1, \dots, f_{m_1}, f_1, \dots, f_{m_2})}((t_1, \dots, t_m, \varepsilon u_1, \dots, \varepsilon u_{m_1}, \varepsilon' v_1, \dots, \varepsilon' v_{m_2})) \\ &\quad - L_{f,(f_1, \dots, f_m, f_1, \dots, f_{m_1}, f_1, \dots, f_{m_2})}((t_1, \dots, t_m, 0, \dots, 0)) \end{aligned}$$

so that the differentiability of  $L_{f,(f_1, \dots, f_m, f_1, \dots, f_{m_1}, f_1, \dots, f_{m_2})}$  at  $(t_1, \dots, t_m, 0, \dots, 0)$  implies the differentiability of  $L_{f+\sum_{k=1}^m t_k f_k, (\sum_{k=1}^{m_1} u_k f_k, \sum_{k=1}^{m_2} v_k f_k)}$  at 0; the conclusion follow from Lemma 6.  $\square$

For each  $f \in C(\Omega)$  let  $Q_f$  be the map defined on  $C(\Omega)$  by

$$\forall g \in C(\Omega), \quad Q_f(g) = P^\tau(f + g) - P^\tau(f).$$

**Lemma 8.** *For each  $f \in C(\Omega)$  the function  $Q_f$  is proper convex continuous and its Legendre-Fenchel transform  $Q_f^*$  fulfils*

$$\forall \mu \in \widetilde{\mathcal{M}}(\Omega), \quad Q_f^*(\mu) = \begin{cases} P^\tau(f) - h^\tau(\mu) - \mu(f) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \widetilde{\mathcal{M}}(\Omega) \setminus \mathcal{M}^\tau(\Omega). \end{cases}$$

*In particular,  $Q_f^*$  vanishes exactly on the set of equilibrium states for  $f$ .*

*Proof.* Let  $f \in C(\Omega)$ . Clearly,  $Q_f$  is proper convex and continuous since  $P^\tau$  and  $\widehat{f}$  are (cf. §2.1 and §2.2). Putting

$$U(\mu) = \begin{cases} -\mu(f) - h^\tau(\mu) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \widetilde{\mathcal{M}}(\Omega) \setminus \mathcal{M}^\tau(\Omega), \end{cases}$$

we have

$$P^\tau(f + g) = \sup_{\mu \in \mathcal{M}^\tau(\Omega)} \{\mu(f + g) + h^\tau(\mu)\} = \sup_{\mu \in \widetilde{\mathcal{M}}(\Omega)} \{\mu(g) - U(\mu)\}.$$

Since  $h^\tau$  is bounded affine and upper semi-continuous (cf. §2.1),  $U$  is proper convex and lower semi-continuous; consequently, we have  $U = U^{**}$  (cf. §2.2) *i.e.*

$$\forall \mu \in \widetilde{\mathcal{M}}(\Omega), \quad U(\mu) = \sup_{g \in C(\Omega)} \{\mu(g) - P^\tau(f + g)\} = \sup_{g \in C(\Omega)} \{\mu(g) - P^\tau(f) - Q_f(g)\} =$$

$$-P^\tau(f) + \sup_{g \in C(\Omega)} \{\mu(g) - Q_f(g)\} = -P^\tau(f) + Q_f^*(\mu),$$

which proves the lemma.  $\square$

For each  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$  let  $I_{f, (f_1, \dots, f_n)}$  be the function defined on  $\mathbb{R}^n$  by

$$\forall x \in \mathbb{R}^n, \quad I_{f, (f_1, \dots, f_n)}(x) = \inf\{Q_f^*(\mu) : \mu \in \mathcal{M}(\Omega), (\mu(f_1), \dots, \mu(f_n)) = x\}.$$

Since  $\mathcal{M}^\tau(\Omega)$  is compact and  $Q_f^*$  is lower semi-continuous,  $I_{f, (f_1, \dots, f_n)}(x)$  is a minimum if and only if  $x \in (\widehat{f_1}, \dots, \widehat{f_n})(\mathcal{M}^\tau(\Omega))$  by Lemma 8, hence

$$I_{f, (f_1, \dots, f_n)}(x) = \begin{cases} Q_f^*(\mu_x) \text{ for some } \mu_x \in \mathcal{M}^\tau(\Omega) & \text{if } x \in (\widehat{f_1}, \dots, \widehat{f_n})(\mathcal{M}^\tau(\Omega)) \\ +\infty & \text{if } x \notin (\widehat{f_1}, \dots, \widehat{f_n})(\mathcal{M}^\tau(\Omega)). \end{cases} \quad (9)$$

**Lemma 9.** *The function  $I_{f, (f_1, \dots, f_n)}$  is proper convex and lower semi-continuous for all  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$ .*

*Proof.* Let  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$ , let  $x \in \mathbb{R}^n$ , let  $(x_i)$  be a net in  $\mathbb{R}^n$  converging to  $x$ , and assume that  $\liminf I_{f, (f_1, \dots, f_n)}(x_i) < \delta$  for some real  $\delta$ . There exists a subnet  $(x_j)$  of  $(x_i)$  such that eventually  $I_{f, (f_1, \dots, f_n)}(x_j) < \delta$  and thus  $Q_f^*(\mu_j) < \delta$  for some  $\mu_j \in \mathcal{M}^\tau(\Omega)$  satisfying  $(\mu_j(f_1), \dots, \mu_j(f_n)) = x_j$ . Let  $(\mu'_j)$  be a subnet of  $(\mu_j)$  converging to some  $\mu' \in \mathcal{M}^\tau(\Omega)$ ; note that  $(\mu'(f_1), \dots, \mu'(f_n)) = x$ . We get

$$I_{f, (f_1, \dots, f_n)}(x) \leq Q_f^*(\mu') \leq \liminf Q_f^*(\mu'_j) < \delta,$$

which proves the lower semi-continuity of  $I_{f,f_1,\dots,f_n}$ . For each  $(x_1, x_2, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \times ]0, 1[$  we have

$$\begin{aligned} I_{f,(f_1,\dots,f_n)}(\beta x_1 + (1-\beta)x_2) &= \inf\{Q_f^*(\mu) : \mu \in \mathcal{M}(\mu), (\mu(f_1), \dots, \mu(f_n)) = \beta x_1 + (1-\beta)x_2\} \\ &\leq \inf\{Q_f^*(\beta\mu_1 + (1-\beta)\mu_2) : (\mu_1, \mu_2) \in \mathcal{M}(\Omega)^2, (\mu_1(f_1), \dots, \mu_n(f_n)) = x_1, \\ &\quad (\mu_2(f_1), \dots, \mu_2(f_n)) = x_2\} \\ &\leq \inf\{\beta Q_f^*(\mu_1) + (1-\beta)Q_f^*(\mu_2) : (\mu_1, \mu_2) \in \mathcal{M}(\Omega)^2, (\mu_1(f_1), \dots, \mu_n(f_n)) = x_1, \\ &\quad (\mu_2(f_1), \dots, \mu_2(f_n)) = x_2\} \leq \beta I_{f,(f_1,\dots,f_n)}(x_1) + (1-\beta)I_{f,(f_1,\dots,f_n)}(x_2), \end{aligned}$$

hence  $I_{f,f_1,\dots,f_n}$  is convex;  $I_{f,(f_1,\dots,f_n)}$  is proper by (9).  $\square$

**Lemma 10.** *We have  $I_{f,(f_1,\dots,f_n)} = L_{f,(f_1,\dots,f_n)}^*$  for all  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$ .*

*Proof.* Let  $((f, f_1, \dots, f_n), n) \in C(\Omega)^{n+1} \times \mathbb{N}$  and let  $\langle \cdot, \cdot \rangle$  denote the scalar product in  $\mathbb{R}^n$ . Suppose there exists  $t = (t_1, \dots, t_n) \in \mathbb{R}^n$  such that

$$\sup_{x \in \mathbb{R}^n} \{\langle t, x \rangle - I_{f,(f_1,\dots,f_n)}(x)\} < L_{f,(f_1,\dots,f_n)}(t).$$

Since

$$L_{f,(f_1,\dots,f_n)}(t) = Q_f \left( \sum_{k=1}^n t_k f_k \right) = Q_f^{**} \left( \sum_{k=1}^n t_k f_k \right) = \sup_{\mu \in \mathcal{M}^\tau(\Omega)} \{\langle t, (\mu(f_1), \dots, \mu(f_n)) \rangle - Q_f^*(\mu)\}$$

by Lemma 8, there exists  $\mu \in \mathcal{M}^\tau(\Omega)$  such that

$$\sup_{x \in \mathbb{R}^n} \{\langle t, x \rangle - I_{f,(f_1,\dots,f_n)}(x)\} < \langle t, (\mu(f_1), \dots, \mu(f_n)) \rangle - Q_f^*(\mu),$$

which gives the contradiction by taking  $x = (\mu(f_1), \dots, \mu(f_n))$  in the left hand side; therefore, we have

$$\forall t \in \mathbb{R}^n, \quad I_{f,(f_1,\dots,f_n)}^*(t) \geq L_{f,(f_1,\dots,f_n)}(t). \quad (10)$$

If  $\sup_{x \in \mathbb{R}^n} \{\langle t, x \rangle - I_{f,(f_1,\dots,f_n)}(x)\} > L_{f,(f_1,\dots,f_n)}(t)$  for some  $t \in \mathbb{R}^n$ , then

$$\langle t, x \rangle - I_{f,(f_1,\dots,f_n)}(x) > \sup_{\mu \in \mathcal{M}^\tau(\Omega)} \{\langle t, (\mu(f_1), \dots, \mu(f_n)) \rangle - Q_f^*(\mu)\}$$

for some  $x = (\mu_x(f_1), \dots, \mu_x(f_n))$  with  $\mu_x \in \mathcal{M}^\tau(\Omega)$  fulfilling  $I_{f,(f_1,\dots,f_n)}(x) = Q_f^*(\mu_x)$ , which gives the contradiction; consequently, we have  $I_{f,(f_1,\dots,f_n)}^*(t) \leq L_{f,(f_1,\dots,f_n)}(t)$  for all  $t \in \mathbb{R}^n$ , which together with (10) yields  $I_{f,(f_1,\dots,f_n)}^* = L_{f,(f_1,\dots,f_n)}$ ; the conclusion follows from the foregoing equality together with Lemma 9.  $\square$

**Lemma 11.**  *$C(\Omega)$  contains an element admitting several equilibrium sates.*

*Proof.* Since  $\mathcal{M}^\tau(\Omega)$  is not a singleton, Theorem 3.4 of [9] ensures the existence of some  $a \in A(\mathcal{M}^\tau(\Omega))$  where  $P_{-h\tau}$  is not Gateaux differentiable. Since the map  $C(\Omega) \ni f \mapsto \widehat{f} \in A(\mathcal{M}^\tau(\Omega))$  is surjective, there exists  $f \in C(\Omega)$  such that  $\widehat{f} = a$ . By the equations (18) in Appendix A the function  $P^\tau$  is not Gateaux differentiable at  $f$ ; equivalently,  $f$  has several equilibrium sates (cf. §2.2).  $\square$

**Lemma 12.** *Let  $W$  be a  $\sigma$ -compact vector space dense in  $C(\Omega)$ . Assume that property (i) of Theorem 1 holds. There exists a  $\sigma$ -compact infinite dimensional vector space  $\widetilde{W}$  linearly independent from  $W$  such that  $f + h$  has a unique equilibrium state for all  $(f, h) \in W \times (\widetilde{W} \setminus \{0\})$ .*

*Proof.* (We use the notations and results of Appendix A.) By Lemma 11 there exists  $f_0 \in C(\Omega)$  admitting several equilibrium states. Put  $W_0 = W + \text{span}(\{f_0\})$ . For each  $n \in \mathbb{N}$  let  $\mathcal{P}(n)$  denote the following property: There exists a  $n$ -dimensional vector space  $W_n$  linearly independent from  $W_0$  such that  $f + h$  has a unique equilibrium state for all  $(f, h) \in W_0 \times (W_n \setminus \{0\})$ . Put  $S = \{\widehat{f} : f \in W_0\}$  so that  $S$  is a  $\sigma$ -compact vector space dense in  $A(\mathcal{M}^\tau(\Omega))$ . Let  $G_1$  be the subset of  $A(\mathcal{M}^\tau(\Omega))$  as in Theorem 3 of Appendix A for the above choice of  $S$ ; we have  $G_1 \supset \{\widehat{f} + ta_1 : f \in W_0, t \in \mathbb{R} \setminus \{0\}\}$  for some  $a_1 \in A(\mathcal{M}^\tau(\Omega))$ . Let  $h_1 \in C(\Omega)$  such that  $\widehat{h}_1 = a_1$ . The equations (18) in Appendix A shows that  $P_{-h}^\tau$  is Gateaux differentiable at  $\widehat{f} + ta_1$  if and only if  $P^\tau$  is Gateaux differentiable at  $f + th_1$  for all  $(f, t) \in W_0 \times \mathbb{R} \setminus \{0\}$ . Note that  $h_1 \notin W_0$  (because  $P^\tau$  is not Gateaux differentiable at  $f_0$ ). Therefore,  $\mathcal{P}(1)$  holds by putting  $W_1 = \text{span}(\{h_1\})$ . Assume that  $\mathcal{P}(n)$  holds for  $n \in \mathbb{N}$ . Put  $W' = W_0 \oplus W_n$ . Since  $W'$  is a  $\sigma$ -compact vector space dense in  $C(\Omega)$  and containing  $f_0$ , the conclusion of the preceding case with  $W'$  in place of  $W_0$  and  $n = 1$  ensures the existence of some  $h_{n+1} \in C(\Omega) \setminus W'$  such that that  $f + th_{n+1}$  has a unique equilibrium state for all  $(f, t) \in W' \times \mathbb{R} \setminus \{0\}$ . Therefore,  $\mathcal{P}(n+1)$  holds by putting  $W_{n+1} = W_n + \text{span}(\{h_{n+1}\})$ . Consequently,  $\mathcal{P}(n)$  holds for all  $n \in \mathbb{N}$  and the conclusion follows by putting  $\widetilde{W} = \text{span}(\bigcup_{n \in \mathbb{N}} W_n)$ .  $\square$

**Lemma 13.** *If property (i) of Theorem 1 does not holds, then there exists an integer  $m \geq 2$ ,  $\varepsilon > 0$  and a nonempty open subset  $G$  of  $C(\Omega)^m$  such that for each  $(g_1, \dots, g_m) \in G$  and for each  $g \in C(\Omega) \setminus \text{span}\{g_1, \dots, g_m\}$  with  $\|g\| < \varepsilon$  the function  $P^\tau$  is not Gateaux differentiable at  $g + \sum_{k=1}^m t_k g_k$  for some  $(t_1, \dots, t_m) \in \mathbb{R}^m$ .*

*Proof.* (We use the notations and results of Appendix A.) Let  $n$  be an integer as in Theorem 4 of Appendix A and put  $A_\varepsilon(\mathcal{M}^\tau(\Omega)) = \{a \in A(\mathcal{M}^\tau(\Omega)) : \|a\| < \varepsilon\}$  for all  $\varepsilon > 0$ . The proof of Theorem 4 of Appendix A reveals that  $n \geq 3$  and there is a nonempty open subset  $U_{n-1}$  of  $A(\mathcal{M}^\tau(\Omega))^{n-1}$  and  $\varepsilon > 0$  such that for each  $((a_1, \dots, a_{n-1}), a) \in U_{n-1} \times A_\varepsilon(\mathcal{M}^\tau(\Omega)) \setminus \text{span}(\{a_1, \dots, a_{n-1}\})$  the function  $P_{-h}^\tau$  is not Gateaux differentiable at  $a + \sum_{k=1}^{n-1} t_k a_k$  for some  $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ . Note that when  $A(\mathcal{M}^\tau(\Omega)) = \text{span}(\{a_1, \dots, a_{n-1}\})$ , the above assertion holds trivially by emptiness of the premise (because  $U_{n-1} \times A_\varepsilon(\mathcal{M}^\tau(\Omega)) \setminus \text{span}\{a_1, \dots, a_{n-1}\} = \emptyset$ ). Taking account of the equations (18) in Appendix A together with the continuity and linearity of the map  $C(\Omega) \ni g \mapsto \widehat{g}$ , the conclusion follows by putting  $m = n - 1$  and  $G = \{g \in C(\Omega) : \widehat{g} \in U_m\}$ .  $\square$

**Lemma 14.** *Let  $W$  and  $\widetilde{W}$  be linearly independent infinite dimensional vector subspaces of  $C(\Omega)$ , let  $(f_n)$  (resp.  $(h_n)$ ) be a sequence in  $W$  (resp.  $\widetilde{W}$ ) and let us consider the following inclusion:*

$$\text{span}(\{f_n + h_n : n \in \mathbb{N}\}) \setminus \{0\} \subset W + (\widetilde{W} \setminus \{0\}). \quad (11)$$

If the set  $\{h_n : n \in \mathbb{N}\}$  is linearly independent, then (11) holds. If the set  $\{f_n : n \in \mathbb{N}\}$  is linearly independent and (11) holds, then the set  $\{h_n : n \in \mathbb{N}\}$  is linearly independent.

*Proof.* Suppose that the set  $\{h_n : n \in \mathbb{N}\}$  is linearly independent and (11) does not hold. Since  $\text{span}(\{f_n + h_n : n \in \mathbb{N}\}) \subset W + \widetilde{W}$  there exists  $N \in \mathbb{N}$ ,  $\{c_1, \dots, c_N\} \subset \mathbb{R}$  and  $g \in W$  such that

$$0 \neq \sum_{k=1}^N c_k (f_{n_k} + h_{n_k}) = g \quad (12)$$

hence

$$\sum_{k=1}^N c_k h_{n_k} = g - \sum_{k=1}^N c_k f_{n_k} \in W.$$

Since  $W \cap \widetilde{W} = \{0\}$ , the above expression implies  $\sum_{k=1}^N c_k h_{n_k} = 0$  hence  $c_k = 0$  for all  $k \in \{1, \dots, N\}$  by linearly independence of  $\{h_n : n \in \mathbb{N}\}$ , which contradicts (12).

Conversely, suppose that set  $\{f_n : n \in \mathbb{N}\}$  is linearly independent, (11) holds and the set  $\{h_n : n \in \mathbb{N}\}$  is not linearly independent. Let  $\{h_{n_1}, \dots, h_{n_N}\}$  be a linearly dependent subset of  $\{h_n : n \in \mathbb{N}\}$ . There are real numbers  $c_1, \dots, c_N$  with  $c_{k_0} \neq 0$  for some  $k_0 \in \{1, \dots, N\}$  such that  $\sum_{k=1}^N c_k h_{n_k} = 0$ . The linear independence of  $\{f_n : n \in \mathbb{N}\}$  implies  $\sum_{k=1}^N c_k f_{n_k} \neq 0$  hence

$$0 \neq \sum_{k=1}^N c_k (f_{n_k} + h_{n_k}) = \sum_{k=1}^N c_k f_{n_k},$$

which together with (11) yields

$$\sum_{k=1}^N c_k f_{n_k} \in W \cap (W + (\widetilde{W} \setminus \{0\})).$$

The contradiction follows from the above expression since the hypothesis  $W \cap \widetilde{W} = \{0\}$  implies  $W \cap (W + \widetilde{W} \setminus \{0\}) = \emptyset$ .  $\square$

*Proof of Theorem 1.* Let  $(v_1)$  (resp.  $(vi_2)$ ,  $(vii_3)$ ,  $(iv_4)$ ) denote the statement obtained from  $(v)$  (resp.  $(vi)$ ,  $(vii)$ ,  $(iv)$ ) with the changes described in b)1) (resp. b)2), b)3), b)4)).

- Proof of  $(iv_4) \Rightarrow (iv)$ ,  $(v_1) \Rightarrow (v)$ ,  $(vi_2) \Rightarrow (vi)$  and  $(vii_3) \Rightarrow (vii)$ , with the same space  $V$  in the premise as in the conclusion regarding the last two implications: The first implication is obvious; all the others as well as the clarification on  $V$  follow from Lemma 2 and the observation before Lemma 5.

- Proof of  $(i) \Rightarrow (ii)$ : Assume that  $(i)$  holds. Let  $\mu \in \mathcal{M}^\tau(\Omega)$ . For each  $\varepsilon > 0$  there is a sequence  $(\mu_{\varepsilon,n})$  in  $\mathcal{E}^\tau(\Omega)$  converging to  $\mu$  and fulfilling  $h^\tau(\mu_{\varepsilon,n}) > h^\tau(\mu) - \varepsilon$  for all  $n \in \mathbb{N}$ . Let us consider the product set  $\wp = ]0, +\infty[ \times \mathbb{N}]^{0,+\infty[$  directed as before Theorem 1 (with  $l = 1$ ), let  $(\mu_i)_{i \in \wp}$  be the net defined by putting  $\mu_i = \mu_{\varepsilon,u(i)}$  for all  $i = (\varepsilon, u) \in \wp$  and let  $s$  be the function defined on  $]0, +\infty[ \times \mathbb{N}$  by

$$\forall(\varepsilon, n) \in ]0, +\infty[ \times \mathbb{N}, \quad s(\varepsilon, n) = h^\tau(\mu_{\varepsilon,n}).$$

We have

$$\forall \varepsilon > 0, \quad h^\tau(\mu) \geq \limsup_n h^\tau(\mu_{\varepsilon,n}) \geq \liminf_n h^\tau(\mu_{\varepsilon,n}) \geq h^\tau(\mu) - \varepsilon$$

and letting  $\varepsilon \rightarrow 0$  yields

$$h^\tau(\mu) = \limsup_\varepsilon \limsup_n h^\tau(\mu_{\varepsilon,n}) = \liminf_\varepsilon \liminf_n h^\tau(\mu_{\varepsilon,n})$$

and

$$\lim_\varepsilon \lim_n \mu_{\varepsilon,n} = \mu.$$

The last above expression shows that  $\lim_i \mu_i = \mu$  ([11], Theorem on Iterated Limits, p. 69) and the first one together with Lemma 1 (applied with  $J = ]0, +\infty[$  and  $L = \mathbb{N}$ ) yields  $\lim_i h^\tau(\mu_i) = h^\tau(\mu)$ , which proves (ii).

• Proof of (i)  $\Rightarrow$  (iv<sub>4</sub>): Assume that (i) holds. Let  $\widetilde{W}$  be as in Lemma 12. Let  $(f_n)$  be a Schauder basis of  $C(\Omega)$  included in  $W$  (cf. §2.1) and let  $\{h_n : n \in \mathbb{N}\}$  be a linearly independent subset of  $\widetilde{W}$  fulfilling

$$\sum_{n=1}^{+\infty} \left( \sup_{f \in C(\Omega), \|f\| \leq 1} |\lambda_n(f)| \right) \|h_n\| < 1$$

(with  $\lambda_n(f)$  as in §2.1). The sequence  $(f_n + h_n)$  is a Schauder basis of  $C(\Omega)$  (cf. §2.1) hence  $\text{span}(\{f_n + h_n : n \in \mathbb{N}\})$  is dense in  $C(\Omega)$ ; clearly,  $\text{span}(\{f_n + h_n : n \in \mathbb{N}\})$  is  $\sigma$ -compact; since  $W \cap \widetilde{W} = \{0\}$  and  $\widetilde{W}$  is infinite dimensional,  $\text{span}(\{f_n + h_n : n \in \mathbb{N}\})$  is infinite dimensional and linearly independent from  $W$ ; furthermore, we have

$$\text{span}(\{f_n + h_n : n \in \mathbb{N}\}) \setminus \{0\} \subset W + (\widetilde{W} \setminus \{0\})$$

by Lemma 14; therefore, (iv<sub>4</sub>) holds with  $V = \text{span}(\{f_n + h_n : n \in \mathbb{N}\})$ .

• Proof of (ii)  $\Rightarrow$  (i): Assume that (ii) holds. Let  $r \in \mathbb{R}$ . If  $r \geq \sup_{\mathcal{M}^\tau(\Omega)} h^\tau$  then  $\{\mu \in \mathcal{M}^\tau(\Omega) : h^\tau(\mu) > r\} = \emptyset$  and (i) holds trivially. Assume that  $r < \sup_{\mathcal{M}^\tau(\Omega)} h^\tau$  and let  $\mu \in \mathcal{M}^\tau(\Omega)$  fulfilling  $h^\tau(\mu) > r$ . The condition (ii) applied to the point  $(\mu, h^\tau(\mu))$  of the graph of  $h^\tau$  ensures the existence of a net  $(\mu_i)$  in  $\mathcal{E}^\tau(\Omega)$  converging to  $\mu$  and such that  $\lim h^\tau(\mu_i) = h^\tau(\mu)$ ; thus,  $h^\tau(\mu_i) > r$  eventually, and (i) holds.

• Proof of (ii)  $\Rightarrow$  (v<sub>1</sub>) with the rate function given in a)2): Assume that (ii) holds. Let  $(\nu_\alpha, t_\alpha)$  be a net where  $\nu_\alpha$  is a Borel probability measure on  $\mathcal{M}(\Omega)$ ,  $t_\alpha > 0$  and  $(t_\alpha)$  converges to zero. Assume that

$$\forall g \in C(\Omega), \quad \lim_\alpha t_\alpha \log \int_{\mathcal{M}(\Omega)} e^{t_\alpha^{-1} \int_\Omega g(\omega) \mu(d\omega)} \nu_\alpha(d\mu) = Q_{f_0}(g).$$

Taking account of Lemma 3, the condition (ii) together with the above equality shows that the hypotheses of Theorem 5.2(b) of [3] hold; therefore, the net  $(\nu_\alpha)$  satisfies a large deviation principle in  $\mathcal{M}(\Omega)$  with powers  $(t_\alpha)$  and rate function  $Q_{f_0}^*|_{\mathcal{M}(\Omega)}$ . By Lemma 8, the condition (ii) is equivalent to the density of the graph of  $Q_{f_0}^*|_{\mathcal{E}^\tau(\Omega)}$  in the graph of  $Q_{f_0}^*|_{\mathcal{M}^\tau(\Omega)}$ , which proves (v<sub>1</sub>).

• Proof of  $(v) \Rightarrow (ii)$ : Assume that  $(v)$  holds. Since  $I$  is convex and for each  $g \in C(\Omega)$ ,  $Q_{f_0}(g)$  coincides with the value at  $\hat{g}$  of the limiting log-moment generating function associated with  $(\nu_{f_0, \alpha}^\tau, t_\alpha^\tau)$  by Lemma 2, we have  $I = Q_{f_0|_{\mathcal{M}(\Omega)}}^*$  (cf. §2.3) so that  $(ii)$  follows from Lemma 8.

• The second equality in  $a)2)$  follows from Lemma 10.

We have proved

$$(i) \Leftrightarrow (ii) \Leftrightarrow (v) \Leftrightarrow (v_1) \Rightarrow (iv_4) \Rightarrow (iv) \quad \text{and property } a)2) \text{ concerning } v_1). \quad (13)$$

• Proof of  $(iv) \Rightarrow (iii)$  with  $D_n$  as in  $a)3)$  for all  $n \in \mathbb{N}$ : Assume that  $(iv)$  holds. Let  $V$  as in  $(iv)$ , let  $(f, (g_1, \dots, g_n), n) \in \Sigma \times V^n \times \mathbb{N}$  and let  $\varepsilon > 0$ . Since  $V$  is infinite dimensional and dense in  $C(\Omega)$  there exists  $g_{n+1} \in V \setminus \text{span}\{f, g_1, \dots, g_n\}$  such that  $\|f + g_{n+1}\| < \varepsilon$ . Therefore,  $f + g_{n+1} \notin \text{span}\{g_1, \dots, g_n\}$  and  $f + g_{n+1} + \sum_{k=1}^n t_k g_k$  has a unique equilibrium state for all  $(t_1, \dots, t_n) \in \mathbb{R}^n$ , which proves  $(iii)$  by putting  $D_n = V^n$  and  $g = f + g_{n+1}$ .

• Proof of  $(iii) \Rightarrow (i)$ : Assume that  $(iii)$  holds. Let  $n \in \mathbb{N}$  and let  $\varepsilon > 0$ . The set of all  $(g_1, \dots, g_n) \in C(\Omega)^n$  such that for each  $g \in C(\Omega) \setminus \text{span}\{g_1, \dots, g_n\}$  with  $\|g\| < \varepsilon$  the function  $g + \sum_{k=1}^n t_k g_k$  has several equilibrium states for some  $(t_1, \dots, t_n) \in \mathbb{R}^n$ , has empty interior; since  $g + \sum_{k=1}^n t_k g_k$  has several equilibrium states if and only if  $P_{-h^\tau}$  is not Gateaux differentiable at  $\hat{g} + \sum_{k=1}^n t_k \hat{g}_k$  (by (18) in Appendix A),  $(i)$  follows from Lemma 13.

The last two above implications together with (13) yield

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv_4) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (v_1) \quad \text{and property } a)3) \text{ concerning } iv) \text{ and } iv_4). \quad (14)$$

• Proof of  $(iv) \Rightarrow (vi_2)$  with the rate function given in  $a)2)$ : Assume that  $(iv)$  holds. Let  $V$  as in  $(iv)$ . Since  $(iv) \Leftrightarrow (v_1)$  by (14), and since  $f_0$  in  $(v_1)$  is an arbitrary element of  $C(\Omega)$ ,  $(v_1)$  holds in particular with  $f_0 = f + g$  for all  $(f, g) \in \Sigma \times V$ ; then,  $(vi_2)$  with  $V$  follows from the last assertion of Lemma 8.

• Proof of  $(vi) \Rightarrow (iv)$ : Assume that  $(vi)$  holds. Let  $V$  as in  $(vi)$ . The same argument as in the proof of  $(v) \Rightarrow (ii)$  shows that the rate function associated with  $(\nu_{f+g, \alpha}^\tau, t_\alpha^\tau)$  is  $Q_{f+g|_{\mathcal{M}(\Omega)}}^*$  so that  $(iv)$  with  $V$  follows from the last assertion of Lemma 8.

We have proved

$$(iv) \Leftrightarrow (vi) \Leftrightarrow (vi_2) \quad \text{and property } a)2) \text{ concerning } vi_2). \quad (15)$$

• Proof of  $(iv) \Rightarrow (vii_3)$  with the rate function given in  $a)2)$ : Assume that  $(iv)$  holds. Let  $V$  as in  $(iv)$ . Since  $(i) \Leftrightarrow (iv)$  by (14), and since  $(i)$  does not depend on  $\Sigma$ , one can assume that  $\Sigma = W$ . Let  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$  and let  $(\mu_\alpha, t_\alpha)$  be a net as in  $b)3)$ . By Lemma 5 (applied with  $f + g$  in place of  $f$ ) the function  $L_{f+g, (f_1, \dots, f_n)}$  is differentiable on  $\mathbb{R}^n$  for all  $((f_1, \dots, f_n), n) \in E^n \times \mathbb{N}$  and consequently its Legendre-Fenchel transform  $L_{f+g, (f_1, \dots, f_n)}^*$  is essentially strictly convex (cf. §2.2). By Gärtner's theorem (cf. §2.3) the net  $(\mu_\alpha)$  satisfies a large deviation principle in  $\mathbb{R}^n$  with powers  $(t_\alpha)$  and rate function  $L_{f+g, (f_1, \dots, f_n)}^*$ , which proves  $(vii_3)$  with  $V$ .



• Proof of  $(vii) \Rightarrow (viii)$ : Assume that  $(vii)$  holds. Let  $V$  be as in  $(vii)$  and let  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$ . Lemma 2 implies that  $L_{f+g, (f_1, \dots, f_n)}$  is the limiting log-moment generating function associated with  $((\widehat{f_1}, \dots, \widehat{f_n})[\nu_{f+g, \alpha}^\tau], t_\alpha^\tau)$ . Since the rate function governing the large deviation principle of the net  $((\widehat{f_1}, \dots, \widehat{f_n})[\nu_{f+g, \alpha}^\tau])$  is convex, it must be the Legendre-Fenchel transform  $L_{f+g, (f_1, \dots, f_n)}^*$  of  $L_{f+g, (f_1, \dots, f_n)}$  (cf. §2.3) hence  $(viii)$  holds with  $V$ .

• Proof of  $(viii) \Rightarrow (ix)$  with the same space  $V$  in the premise as in the conclusion: This follows from Lemma 10.

• Proof of  $(ix) \Rightarrow (iv)$ : Assume that  $(ix)$  holds. Let  $V$  be as in  $(ix)$ . For each  $(f, g, (f_1, \dots, f_n), n) \in \Sigma \times V \setminus \{0\} \times E^n \times \mathbb{N}$  the function  $L_{f+g, (f_1, \dots, f_n)}^*$  is essentially strictly convex by Lemma 10 hence  $L_{f+g, (f_1, \dots, f_n)}$  is differentiable on  $\mathbb{R}^n$  (cf. §2.2) and  $(iv)$  with  $V$  follows from Lemma 7.

We have proved

$$(iv) \Leftrightarrow (vii_3) \Leftrightarrow (vii) \Leftrightarrow (viii) \Leftrightarrow (ix) \quad \text{and property a)2) concerning } vii_3). \quad (16)$$

The above proof reveals that the rate function in  $(v_1)$ ,  $(vi_2)$ ,  $(vii_3)$  is given by the Legendre-Fenchel transform of the limiting log-moment generating function associated with the net concerned; since this limiting log-moment generating function is the same in  $(v_1)$ ,  $(vi_2)$ ,  $(vii_3)$  as in  $(v)$ ,  $(vi)$ ,  $(vii)$ , respectively (by Lemma 2 and the observation before Lemma 5) it follows that property a)2) concerning  $v)$ ,  $vi)$ ,  $vii)$  hold.

Taking into account the foregoing observation and putting together (14), (15), (16) proves part b), property a)2) and all the equivalences of part a) as well as property a)3) concerning  $iv)$ ; furthermore, the above proof shows that the same vector space  $V$  may be used in each of the condition appearing in (15) (resp. (16)), which gives property a)1) and property a)3) in full. The proof of part a) is complete.

The first assertion of part c) follows noting that the above proof works verbatim replacing  $V \setminus \{0\}$  by  $V$  when each element of  $\Sigma$  has a unique equilibrium state.

Let  $(iv_0)$ ,  $(vi_0)$ ,  $(vi_{2,0})$ ,  $(vii_0)$ ,  $(vii_{3,0})$ ,  $(viii_0)$  and  $(ix_0)$  denote the conditions obtained respectively from  $(iv)$ ,  $(vi)$ ,  $(vi_2)$ ,  $(vii)$ ,  $(vii_3)$ ,  $(viii)$  and  $(ix)$  replacing  $V \setminus \{0\}$  by  $V$ . The equivalences

$$(iv_0) \Leftrightarrow (vi_0) \Leftrightarrow (vi_{2,0}) \Leftrightarrow (vii_0) \Leftrightarrow (viii_{3,0}) \Leftrightarrow (viii_0) \Leftrightarrow (ix_0)$$

with the same space  $V$  in all the above conditions follows replacing  $V \setminus \{0\}$  by  $V$  in the proof of (15) and (16); this proves the last assertion of part c).  $\square$

*Proof of Theorem 2.* We can assume that  $\Sigma = E = W$  (cf. Remark 1). Let  $\mathcal{N}$  be a nonempty subset of  $\mathbb{N} \cup \{0\}$ .

First assume that  $\mathcal{N} = \mathbb{N} \cup \{0\}$ . For each  $n \in \mathbb{N} \cup \{0\}$  let  $\mathcal{P}(n)$  denotes the following property: There exists a finite sequence  $V_0, \dots, V_n$  of mutually linearly independent  $\sigma$ -compact infinite dimensional vector subspaces of  $C(\Omega)$  such that  $\bigoplus_{j=0}^n V_j$  fulfils the conditions  $(iv)$ ,  $(vi)$ ,  $(vii)$ ,  $(viii)$ ,  $(ix)$  of Theorem 1. We know that  $\mathcal{P}(0)$  holds by property 1) in part a)

of Theorem 1; note that in particular (D) holds. Let  $n \in \mathbb{N} \cup \{0\}$  and assume that  $\mathcal{P}(n)$  holds. Put  $W_n = W \oplus \bigoplus_{j=0}^n V_j$ ; note that  $W_n$  is a  $\sigma$ -compact vector space dense in  $C(\Omega)$  and thus it satisfies the hypotheses of Theorem 1. Since (D) holds, Lemma 12 ensures the existence of a  $\sigma$ -compact infinite dimensional vector space  $\widetilde{W}_n$  linearly independent from  $W_n$  and such that  $f + g$  has a unique equilibrium state for all  $(f, g) \in W_n \times (\widetilde{W}_n \setminus \{0\})$ . (alternatively, the space  $\widetilde{W}_n$  can be obtained applying Theorem 1 with  $W_n$  in place of  $W$ , taking into account property *b*4)). Let  $(f_{n,k})$  be a Schauder basis of  $C(\Omega)$  included in  $W_n$  (cf. §2.1) and let  $\{h_{n,k} : k \in \mathbb{N}\}$  be a linearly independent subset of  $\widetilde{W}_n$  fulfilling

$$\sum_{k=1}^{+\infty} \left( \sup_{f \in C(\Omega), \|f\| \leq 1} |\lambda_{n,k}(f)| \right) \|h_{n,k}\| < 1,$$

where  $(\lambda_{n,k}(f))$  denotes the coordinates of  $f$  in the basis  $(f_{n,k})$ . Put

$$V_{n+1} = \text{span}(\{f_{n,k} + h_{n,k} : k \in \mathbb{N}\}).$$

The sequence  $(f_{n,k} + h_{n,k})$  is a Schauder basis of  $C(\Omega)$  (cf. §2.1) hence  $V_{n+1}$  is dense in  $C(\Omega)$ ; clearly,  $V_{n+1}$  is  $\sigma$ -compact; since  $\widetilde{W}_n \cap W_n = \{0\}$  and  $\widetilde{W}_n$  is infinite dimensional, it follows that  $V_{n+1}$  is infinite dimensional and fulfils  $V_{n+1} \cap W_n = \{0\}$ ; furthermore, Lemma 14 applied with  $W_n$  and  $\widetilde{W}_n$  in place of  $W$  and  $\widetilde{W}$  yields

$$V_{n+1} \setminus \{0\} \subset W_n \oplus (\widetilde{W}_n \setminus \{0\})$$

hence

$$W + V_{n+1} \setminus \{0\} \subset W_n \oplus (\widetilde{W}_n \setminus \{0\}).$$

The recurrence hypothesis at rank  $n$  implies that  $f + g$  has a unique equilibrium state for all  $(f, g) \in W \times \bigoplus_{j=0}^n V_j \setminus \{0\}$ . Since

$$\bigoplus_{j=0}^{n+1} V_j \setminus \{0\} = \bigoplus_{j=0}^n V_j \setminus \{0\} \cup V_{n+1} \setminus \{0\},$$

it follows that the space  $\bigoplus_{j=0}^{n+1} V_j$  fulfils the condition (iv) of Theorem 1, which gives  $\mathcal{P}(n+1)$  by property *a*1) of Theorem 1. Therefore,  $\mathcal{P}(n)$  holds for all  $n \in \mathbb{N}$  hence the infinite direct sum  $\bigoplus_{j=0}^{\infty} V_j$  fulfils the condition (iv) of Theorem 1 since each element of  $\bigoplus_{j=0}^{\infty} V_j$  (resp.  $\bigoplus_{j=0}^{\infty} V_j \setminus \{0\}$ ) belongs to  $\bigoplus_{j=0}^n V_j$  (resp.  $\bigoplus_{j=0}^n V_j \setminus \{0\}$ ) for some  $n \in \mathbb{N}$ ; consequently, property *a*1) of Theorem 1 implies that the conclusion of part *a*) holds when  $\mathcal{N} = \mathbb{N} \cup \{0\}$ .

Let  $\mathcal{N} \neq \mathbb{N} \cup \{0\}$ . Note that  $W + \bigoplus_{n \in \mathbb{N} \cup \{0\} \setminus \mathcal{N}} V_n$  is a  $\sigma$ -compact vector space dense in  $C(\Omega)$  and thus it fulfils the hypotheses of Theorem 1. Since

$$(W + \bigoplus_{n \in \mathbb{N} \cup \{0\} \setminus \mathcal{N}} V_n) + (\bigoplus_{n \in \mathcal{N}} V_n \setminus \{0\}) \subset W + \bigoplus_{n \in \mathbb{N} \cup \{0\}} V_n \setminus \{0\},$$

the preceding case (when  $\mathcal{N} = \mathbb{N} \cup \{0\}$ ) implies that  $\bigoplus_{n \in \mathcal{N}} V_n$  fulfils the condition (iv) of Theorem 1 applied with  $W + \bigoplus_{n \in \mathbb{N} \cup \{0\} \setminus \mathcal{N}} V_n$  in place of  $W$ ; the conclusion of part *a*) when

$\mathcal{N} \neq \mathbb{N} \cup \{0\}$  follows by property a)1) of Theorem 1 (applied with  $W + \bigoplus_{n \in \mathbb{N} \cup \{0\} \setminus \mathcal{N}} V_n$  in place of  $W$ ). We have proved part a) and the first assertion of part b).

If furthermore  $V_0 \cap W = \{0\}$ , then the above proof works verbatim with  $W$  in place of  $W_0$  and taking  $\widetilde{W}_0 = V_0$  (indeed, the choice of  $W + V_0$  and not just  $W$  as general definition of  $W_0$  is made in order to ensure that  $W_0 \cap \widetilde{W}_0 = \{0\}$ ); if  $W$  contains an element admitting several equilibrium states, then  $V_0 \cap W = \{0\}$  by the condition (iv) of Theorem 1; this proves the last assertion of part b).

For each  $n \in \mathbb{N} \cup \{0\}$  let  $\mathcal{V}_{n+1}$  and  $W_n$  be the spaces defined in part c). Put  $\mathcal{V}_0 = V_0$  and for each  $n \in \mathbb{N} \cup \{0\}$  let  $\mathcal{Q}(n)$  denote the following inclusion:

$$W + \sum_{j=0}^n \widetilde{V}_j \subset W + \bigoplus_{j=0}^n \mathcal{V}_j.$$

By definition  $\mathcal{Q}(0)$  holds. Let  $n \in \mathbb{N} \cup \{0\}$  and assume that  $\mathcal{Q}(n)$  holds. Since by definition

$$\widetilde{V}_{n+1} \subset W + \bigoplus_{j=0}^n \widetilde{V}_j + \mathcal{V}_{n+1},$$

we have

$$W + \sum_{j=0}^{n+1} \widetilde{V}_j \subset W + \bigoplus_{j=0}^n \widetilde{V}_j + \mathcal{V}_{n+1};$$

the recurrence hypothesis at rank  $n$  together with the above inclusion yields

$$W + \sum_{j=0}^{n+1} \widetilde{V}_j \subset W + \bigoplus_{j=0}^{n+1} \mathcal{V}_j,$$

which proves  $\mathcal{Q}(n+1)$ . Therefore,  $\mathcal{Q}(n)$  holds for all  $n \in \mathbb{N} \cup \{0\}$ . Let  $n \in \mathbb{N} \cup \{0\}$ . Since  $\mathcal{V}_k \subset \bigoplus_{j=m_k}^{j=m_{k+1}} V_j$  for all  $k \in \mathbb{N}$  with  $\bigoplus_{j=1}^{\infty} V_j$  as in part a), it follows that  $f + g$  has a unique equilibrium state for all  $(f, g) \in W \times (\bigoplus_{j=0}^{n+1} \mathcal{V}_j \setminus \{0\})$  and in particular for all  $(f, g) \in W \times (\bigoplus_{j=0}^n \mathcal{V}_j \oplus \mathcal{V}_{n+1} \setminus \{0\})$ ; this property together with  $\mathcal{Q}(n)$  shows that all the hypotheses of part b) hold replacing  $V_n$  by  $\widetilde{V}_n$  and taking  $\widetilde{W}_n = \mathcal{V}_{n+1}$  for all  $n \in \mathbb{N} \cup \{0\}$ , which gives all the conclusions of part c).  $\square$

*Proof of Corollary 1.* Let  $(v_f)$  denote the statement obtained from (v) with the change described in the property 2) with  $f \in V$ . Let  $(ii_\sigma)$  (resp.  $(iii_\sigma)$ ,  $(iv_\sigma)$ ,  $(v_\sigma)$ ,  $(v_{\sigma,f})$ ) denote the statement obtained from (ii) (resp. (iii), (iv), (v),  $(v_f)$ ) assuming furthermore that  $V$  is  $\sigma$ -compact.

The condition  $(iii_\sigma)$  is equivalent to the condition  $(viii)$  of Theorem 1 with  $\Sigma = \{0\}$ ,  $V = W = E$ ; furthermore, the equalities  $V = W = E$  allows us to replace  $V \setminus \{0\}$  by  $V$ . Therefore, the last assertion of part c) of Theorem 1 yields

$$(i) \Leftrightarrow (ii_\sigma) \Leftrightarrow (iii_\sigma) \Leftrightarrow (iv_\sigma) \Leftrightarrow (v_{\sigma,f}) \Leftrightarrow (v_\sigma) \quad (17)$$

and the validity of the property 2); furthermore, by the property *a*)1) of Theorem 1, if a vector space fulfils one of the conditions  $(ii_\sigma)$ ,  $(iii_\sigma)$ ,  $(iv_\sigma)$ ,  $(v_{\sigma,f})$ ,  $(v_\sigma)$  then it fulfils all of them. Since  $(ii_\sigma)$  (resp.  $(iii_\sigma)$ ,  $(iv_\sigma)$ ,  $(v_{\sigma,f})$ ) is clearly equivalent to  $(ii)$  (resp.  $(iii)$ ,  $(iv)$ ,  $(v_f)$ ) we have proved all the assertions of Corollary 1 except the property 1) (note that the property 2) does not depend on the  $\sigma$ -compactness of  $V$ ).

If an infinite dimensional vector space  $V$  dense in  $C(\Omega)$  fulfils one of the conditions  $(ii)$ ,  $(iii)$ ,  $(iv)$ ,  $(v)$ , then for each  $\sigma$ -compact vector space  $V_0$  dense in  $V$  the hypotheses of Theorem 2 hold with  $E = \Sigma = W = V_0$  (cf. Remark 3); therefore, property 1) follows from part *a*) of Theorem 2.  $\square$

#### APPENDIX A.

The paper [9] deals with a nonempty convex compact subset  $K$  of some Hausdorff locally convex real topological vector space, the set  $A(K)$  of all real-valued affine continuous functions on  $K$  endowed with the uniform topology, a convex bounded non-positive-valued lower semi-continuous function  $l$  on  $K$  and the function  $P_l$  on  $A(K)$  associated to  $l$  in the following way:

$$\forall a \in A(K), \quad P_l(a) = \sup_{x \in K} \{a(x) - l(x)\};$$

the map  $P_l$  is real-valued convex and continuous ([9], Proposition 2.3). When  $K = \mathcal{M}^\tau(\Omega)$  it is known that the map  $C(\Omega) \ni g \mapsto \widehat{g} \in A(\mathcal{M}^\tau(\Omega))$  is surjective ([9], Proposition 2.1); when furthermore  $l = -h^\tau$  we have

$$\forall (f, g, t) \in C(\Omega)^2 \times \mathbb{R}, \quad P_{-h^\tau}(\widehat{f + tg}) = P_{-h^\tau}(\widehat{f} + t\widehat{g}) = P^\tau(f + tg); \quad (18)$$

in particular,  $dP^\tau(f; g)$  coincides with the directional derivative of  $P_{-h^\tau}$  at  $\widehat{f}$  in the direction  $\widehat{g}$ . Since  $h^\tau$  is affine and  $\mathcal{E}^\tau(\Omega)$  is the set of extreme points of  $K$ , Theorem 3.2 and Theorem 3.3 of [9] take respectively the following forms:

**Theorem 3. (Israel-Phelps)** *Assume that property (i) of Theorem 1 holds. Then, for each  $\sigma$ -compact subset  $S$  of  $A(\mathcal{M}^\tau(\Omega))$  and for each  $n \in \mathbb{N}$ , the set  $G_n$  of all elements  $(a_1, \dots, a_n) \in A(\mathcal{M}^\tau(\Omega))^n$  such that  $P_{-h^\tau}$  is Gateaux differentiable at  $b + \sum_{k=1}^n t_k a_k$  for all  $(b, (t_1, \dots, t_n)) \in S \times \mathbb{R}^n \setminus \{0\}$ , is a  $G_\delta$  set dense in  $A(\mathcal{M}^\tau(\Omega))^n$ .*

**Theorem 4. (Israel-Phelps)** *If property (i) of Theorem 1 does not hold, then there exists  $n \in \mathbb{N}$  such that the set of all  $(a_1, \dots, a_n) \in A(\mathcal{M}^\tau(\Omega))^n$  for which  $P_{-h^\tau}$  is not Gateaux differentiable at  $\sum_{k=1}^n t_k a_k$  for some  $(t_1, \dots, t_n) \in \mathbb{R}^n \setminus \{0\}$ , has nonempty interior.*

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